

Market Liquidity and Funding Liquidity*

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Abstract

We provide a model that links an assets' market liquidity — i.e., the ease of trading it — and traders' funding liquidity — i.e., their availability of funds. Traders provide market liquidity and their ability to do so depends on their funding. Conversely, traders' funding, i.e., their capital and the margins they are charged, depend on the assets' market liquidity. We show that under certain conditions margins are destabilizing and market liquidity and funding liquidity are mutually reinforcing, leading to liquidity spirals. The model explains the empirically documented features that market liquidity (i) is fragile, i.e. can suddenly dry up, (ii) has commonality across securities, (iii) is related to volatility, (iv) experiences “flight to liquidity” events, and (v) comoves with the market.

Keywords: Liquidity Risk Management, Liquidity, Liquidation, Systemic Risk, Leverage, Margins, Haircuts, Value-at-Risk, Counterparty Credit Risk

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Trading requires capital. When a trader — e.g. a dealer, hedge fund, or investment bank — buys a security, he can use the security as collateral and borrow against it, but he cannot borrow the entire price. The difference between the security’s price and collateral value, denoted the margin, must be financed with the trader’s own capital. Similarly, shortselling requires capital in the form of a margin; it does not free up capital. Hence, at any time the total margins on all positions cannot exceed the trader’s capital.

Our model shows that the funding of traders affects, and is affected by, market liquidity in a profound way. When funding liquidity is tight, traders become reluctant to take on positions, especially “capital-intensive” positions in high-margin securities. This lowers market liquidity. Further, under certain conditions, low future market liquidity increases the risk of financing a trade, thus increasing the margins.

Based on the links between funding and market liquidity, we provide a unified explanation for the main empirical features of market liquidity. In particular, the model implies that market liquidity (i) can suddenly dry up, (ii) has commonality across securities, (iii) is related to volatility, (iv) experiences “flight to liquidity” events, and (v) comoves with the market. The model further has several new testable implications that link margins and dealer funding to market liquidity.

Our model is similar in spirit to Grossman and Miller (1988) with the added feature that speculators face the real-world funding constraint discussed above. In our model, different customers have offsetting demand shocks, but arrive sequentially to the market. This creates a temporary order imbalance. A group of speculators smooth price fluctuations, thus providing market liquidity. Speculators finance their trade through collateralized borrowing from financiers who set the margins to control their value-at-risk (VaR). We derive the competitive equilibrium of the model and explore its liquidity implications. We define market liquidity as the difference between the transaction price and the fundamental value, and funding liquidity as a speculator’s scarcity (or shadow cost) of capital.

We first analyze the properties of margins. We show that margins stabilize the price and decrease with market illiquidity if financiers know that prices diverge due to temporary market illiquidity and know that liquidity will be improved shortly as complementary customers arrive. This is because a current price divergence from fundamentals provide a “cushion” against future price moves, making the speculator’s position less risky in this case.

Margins can increase in market illiquidity and become destabilizing, on the other hand, when financiers’ are unsure whether price changes are due to fundamental news or liquidity shocks and fundamentals have time-varying volatility. This happens when a liquidity shock leads to price volatility which raises the financier’s expectation about future volatility. We predict that speculators face more destabilizing margins in specialized markets in which financiers cannot easily distinguish fundamental shock from liquidity shocks or cannot predict when a trade converges. Figure 1 shows an increase

in margins for S&P 500 futures during the liquidity crises of 1987, 1990, and 1998, and, also, anecdotal evidence from prime brokers suggest that destabilizing margins are empirically relevant.

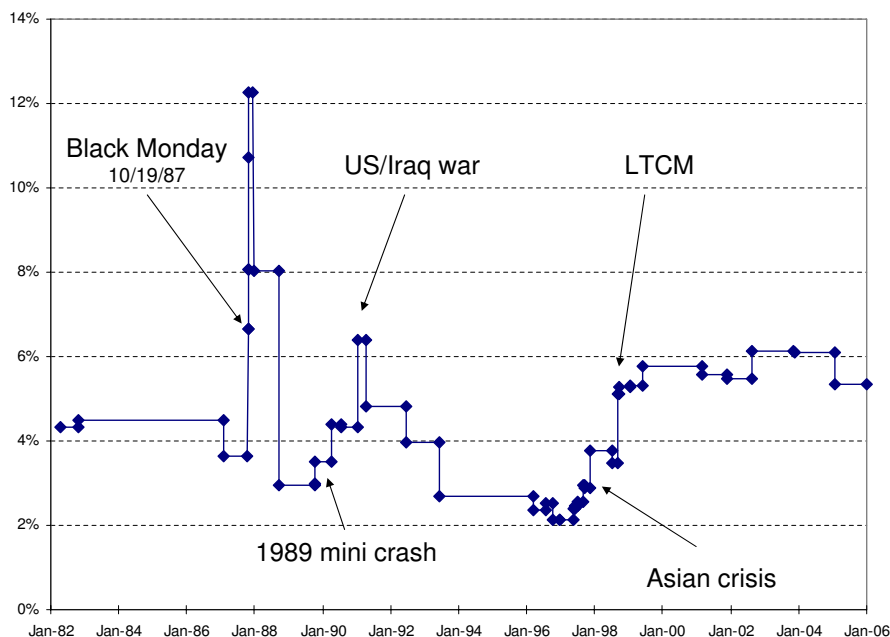


Figure 1: **Margins for S&P500 Futures.** The figure shows margin requirements on S&P500 futures for members of the Chicago Mercantile Exchange as a fraction of the value of the underlying S&P500 index multiplied by the size of the contract. (Initial or maintenance margins are the same for members.) Each dot represents a change in the dollar margin.

Turning to the implications for market liquidity, we first show that, as long as speculator capital is so abundant that there is no risk of hitting the funding constraint, market liquidity is naturally at its highest level and insensitive to marginal changes in capital and margins. However, when speculators hit their capital constraints — or risk hitting their capital constraints over the life of a trade — then they reduce their positions and market liquidity declines.

When margins are destabilizing or the speculators have large existing positions, there can be multiple equilibria and liquidity can be fragile. In one equilibrium, markets are liquid, leading to favorable margin requirements for speculators, which in turn helps speculators make markets liquid. In another equilibrium, markets are illiquid, resulting in larger margin requirements (or speculator losses), thus restricting speculators from providing market liquidity. Importantly, any equilibrium selection has the property that small speculator losses can lead to a discontinuous drop of market liquidity. This “sudden dry-up” or fragility of market liquidity is due to the fact that with high speculator capital, markets must be in the liquid equilibrium, and, if speculator capital is reduced enough, the market must eventually switch to the low-liquidity/high-margin

equilibrium.¹ The events following the Russian default in 1998 is a vivid example of fragility of liquidity since a relatively small shock had a large impact. Indeed, compared to the total market capitalization of the US stock and bond market, the losses due to the Russian default were minuscule, but caused a shiver in world financial markets, see e.g. Figure 1.

Further, when markets are illiquid, market liquidity is highly sensitive to further changes in funding conditions. This is due to two liquidity spirals: first, a “margin spiral” emerges if margins are increasing in market illiquidity because a reduction in speculator wealth lowers market liquidity, leading to higher margins, tightening speculators’ funding constraint further, and so on. For instance, Figure 1 shows how margins gradually escalated within few days after the Black Monday in 1987. Second, a “loss spiral” arises if speculators hold a large initial position that is negatively correlated with customers’ demand shock. In this case, a funding shock increases market illiquidity, leading to speculator losses on their initial position, forcing speculators to sell more, causing a further price drop, and so on.² These liquidity spirals reinforce each other, implying a larger total effect than the sum of their separate effects. Paradoxically, the liquidity spirals imply that a larger shock to the customers’ demand for immediacy leads to a reduction in the provision of immediacy in such stress times.

Our model also provides a natural explanation for the commonality of liquidity across assets since shocks to speculators’ funding constraint affect all securities. This may help explain why market liquidity is correlated across stocks (Chordia, Roll, and Subrahmanyam (2000), Hasbrouck and Seppi (2001) and Huberman and Halka (2001)), and across stocks and bonds (Chordia, Sarkar, and Subrahmanyam (2005)). In support of the idea that commonality is driven at least in part by our funding-liquidity mechanism, Chordia, Sarkar, and Subrahmanyam (2005) find that “money flows ... account for part of the commonality in stock and bond market liquidity.” Moreover, their finding that “during crisis periods, monetary expansions are associated with increased liquidity” is consistent with our model’s prediction that the effects are largest when traders are near their constraint. Coughenour and Saad (2004) provide further evidence of the funding-liquidity mechanism by showing that the comovement in liquidity among stocks handled by the same NYSE specialist firm is higher than for other stocks, commonality is higher for specialists with less capital, and decreases after a merger of specialists.

Next, our model predicts that market liquidity declines as fundamental volatility

¹Fragility can also be caused by asymmetric information on the amount of trading by portfolio insurance traders (Gennotte and Leland (1990)), and by losses on existing positions (Chowdhry and Nanda (1998)).

²The loss spiral is related to the multipliers that arise in Grossman (1988), Kiyotaki and Moore (1997), Shleifer and Vishny (1997), Chowdhry and Nanda (1998), Xiong (2001), Kyle and Xiong (2001), Gromb and Vayanos (2002), Morris and Shin (2004), Plantin, Sapra, and Shin (2005) and others. To our knowledge, our paper is the first to model the margin spiral and the interaction between the two multipliers.

increases, which is consistent with the empirical findings of Benston and Hagerman (1974) and Amihud and Mendelson (1989).³ The model implies that the liquidity differential between high-volatility and low-volatility securities increases as speculator capital deteriorates — a phenomenon often referred to as “flight to quality” or “flight to liquidity.” According to our model, this happens because a reduction in speculator capital induces traders to provide liquidity mostly in securities that do not use much capital (low volatility stocks since they have lower margins). Hence, illiquid securities are predicted to have more liquidity risk. Recently, Hendershott, Moulton, and Seasholes (2006) test these predictions using inventory positions of NYSE specialists as a proxy for funding liquidity. Their finding supports our hypotheses that market liquidity of high volatility stocks is more sensitive to changes in inventory shocks and that this is more pronounced at times of low funding liquidity. Moreover, Pastor and Stambaugh (2003) and Acharya and Pedersen (2005) document empirical evidence consistent with flight to liquidity and the pricing of this liquidity risk.

Market-making firms are often net long the market. For instance, Ibbotson (1999) reports that security brokers and speculators have median market betas in excess of one. Therefore, capital constraints are more likely to be hit during market downturns, and this, together with mechanism outlined in our model, helps explain why sudden liquidity dry-ups occur more often when markets decline and liquidity comoves more during downturns. Following our model’s prediction, Hameed, Kang, and Viswanathan (2005) document that comovements in liquidity indeed are higher during large negative market moves.

Finally, the risk alone that the funding constraint becomes binding limits speculators’ market liquidity provision. Our analysis shows that speculators’ optimal (funding) risk management policy is to maintain a “safety buffer.” This affects initial prices, which increase in the covariance of future prices with future shadow costs of capital (i.e. with future funding illiquidity).

Our paper is related to several literatures.⁴ Most directly related are the mod-

³The link between volatility and liquidity is shared by the models of Stoll (1978), Grossman and Miller (1988), and others. What sets our theory apart is that this link is connected with margin constraints. This leads to testable differences since, according to our model, the link is stronger when speculators are poorly financed, and high-volatility securities are more affected by speculator wealth shocks — our explanation of flight to quality.

⁴Market liquidity is studied specifically in market microstructure Stoll (1978), Ho and Stoll (1981, 1983), (Kyle (1985), Glosten and Milgrom (1985), Grossman and Miller (1988)), and related to the limits of arbitrage (DeLong, Shleifer, Summers, and Waldmann (1990), Shleifer and Vishny (1997), Abreu and Brunnermeier (2002)), funding liquidity is examined in corporate finance (Shleifer and Vishny (1992), Holmström and Tirole (1998,2001)) and banking (Diamond and Dybvig (1983), Allen and Gale (1998, 2004, 2005)), funding and collateral constraints are also studied in macro economics (Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Lustig (2004)), and general equilibrium with incomplete markets (Geanakoplos (1997, 2003)), and, finally recent papers consider illiquidity with constrained traders (Attari, Mello, and Ruckes (2005), Bernardo and Welch (2004), Brunnermeier and Pedersen (2005), Morris and Shin (2004), and Weill (2004)).

els with margin-constrained traders: Grossman and Vila (1992) and Liu and Longstaff (2004) derive optimal strategies in a partial equilibrium with a single security, Chowdhry and Nanda (1998) focus on fragility due to dealers losses, and Gromb and Vayanos (2002) derive a general equilibrium with one security and study welfare and liquidity provision. Our paper contributes to the literature by considering the simultaneous effect of margin constraints on multiple securities and by studying the nature of the margin constraint. Said simply, the existing theoretical literature uses a fixed or decreasing margin constraint — say \$5,000 per contract — and studies what happens when trading losses cause agents to hit this constraint, whereas we study how market conditions lead to changes in the margin requirement itself — e.g. an increase from \$5,000 to \$15,000 per futures contract as happened in October 1987 — and the feedback effects between the margin and the market conditions.

We proceed as follows. First, we describe the institutional features associated with how trades are funded for the main liquidity providers, namely market makers, banks, and hedge funds (Section 1). Then, we describe the model (Section 2) and derive our four main new results are that: (i) margins increase with market illiquidity when financiers cannot distinguish fundamental shocks and liquidity shocks and fundamentals have time-varying volatility (Section 3), (ii) this makes margins destabilizing, leading to sudden liquidity dry ups and margin spirals (Section 4), (iii) liquidity crisis simultaneously affect many securities, mostly risky high-margin securities, resulting in commonality of liquidity and flight to quality (Section 5), and (iv) liquidity risk matters even before speculators hit their capital constraints (Section 6). Finally, we outline how our model's new testable predictions may be helpful for a novel line of empirical work that links measures of speculators' funding conditions to measures of market liquidity (Section 7).

1 Margins, Haircuts & Capital Constraints

A central element of our paper is the capital constraints that the main providers of market liquidity face. To set the stage for our model, we review these institutional features for securities firms such as hedge funds, banks' proprietary traders, and market makers.

1.1 Funding Requirements for Hedge Funds

We first consider the funding issues faced by hedge funds since they have relatively simple balance sheets and face little regulation. A hedge fund's capital consists of its equity capital supplied by the partners and of possible long-term debt financing that can be relied upon during a potential funding crisis. Since a hedge fund is a partnership, the equity is not locked into the firm indefinitely as in a corporation. The

investors (that is, the partners) can withdraw their capital at certain times, but — to ensure funding — the withdrawal is subject to so-called lock-up periods (typically at least a month, often several months or even years). A hedge fund usually cannot issue long-term unsecured bonds, but some (large) hedge funds manage to obtain debt financing in the form of medium-term bank loans or in the form of a guaranteed line of credit.⁵ Hedge funds lever their capital using collateralized borrowing financed by the hedge fund’s prime broker(s). The prime brokerage business is opaque since the terms of the financing are subject to negotiation and are secret to outsiders. We describe stylized financing terms and, later, we discuss caveats.

If a hedge fund buys at time t a long position of $x_t^j > 0$ shares of a security j at price p_t^j , it has to come up with $x_t^j p_t^j$ dollars. The security can, however, be used as collateral for a new loan of, say, l_t^j dollars. The difference between the price of the security and the collateral value is denoted the margin requirement $m_t^{j+} = p_t^j - l_t^j$. Hence, this position uses $x_t^j m_t^{j+}$ dollars of the fund’s capital. The collateralized funding implies that the capital use depends on margins, not notional amounts.

The margins on fixed income securities and over-the-counter (OTC) derivatives are set through a negotiation between the hedge fund and the prime broker that finances the trade. The margins are typically set such as to make the loan almost risk free for the broker, that is, such that it covers the largest possible adverse price move with a certain confidence (i.e. covers the Value-at-Risk).⁶

If the hedge fund wants to sell short a security, $x^j < 0$, then the fund asks one of its brokers to locate a security that can be borrowed, and then the fund sells the borrowed security. Duffie, Gârleanu, and Pedersen (2002) describe in detail the institutional arrangements of shorting. The broker keeps the proceeds of the short sale to be able to repurchase the security if the hedge fund fails and, additionally, requires that the hedge fund posts a margin m_t^{j-} that covers the largest possible adverse price move with a certain confidence.

In the U.S., margins on equities are subject to Regulation T, which stipulates that non-brokers/speculators must have an initial margin (downpayment) of 50% of the market value of the underlying stock, both for new long and short positions. Hedge funds can circumvent Regulation T for instance by organizing the transaction as a total return swap, which is a derivative that is functionally equivalent to buying the stock.

The margin on exchange traded futures (or options) is also set by the exchange. The principle for setting the margin for futures or options is the same as that described

⁵A line of credit may have a “material adverse change” clause or other covenants subject to discretionary interpretation of the lender. Such covenants imply that the line of credit may not be a reliable source of funding during a crisis.

⁶The explicit equation for the margin is given by (6) in Section 2. Often brokers also take into account the time it takes between a fail by the hedge fund is noticed and the security is actually sold. Hence, the margin of a one-day collateralized loan depends on the estimated risk of holding the asset over a time period that is often set to be five to ten days.

above. The margin is set such as to make the exchange almost immune to default risk of the counterparty, and hence riskier contracts have larger margins.

A hedge fund must finance all of its positions, that is, the sum of all the margin requirements on long and short positions cannot exceed the hedge fund's capital. In our model, this is captured by key Equation (4) in Section 2.

At the end of the financing period, time $t + 1$, the position is “marked-to-market,” which means that the hedge fund receives any gains (or pays any losses) that have occurred between t and $t + 1$, that is, the fund receives $x_t^j(p_{t+1}^j - p_t^j)$ and pays interest on the loan at the funding rate. If the trade is kept on, the broker keeps the margin to protect against losses going forward from time $t + 1$. The margin can be adjusted if the risk of the collateral has changed unless the counterparties have contractually fixed the margin for a certain period. Since a hedge fund must be able to finance all its security positions at any point of time, they face the capital constraint (4) at any point in time. Stock exchanges and self-regulatory organizations (e.g. NASD) also impose maintenance/continuation margins for existing stock positions. For example, the NYSE and the NASD require that investors maintain a minimum margin of 25% for long stock positions and of 30% for short stock positions.

Instead of posting risk-free assets (cash), a hedge fund can also post risky assets to cover his margin one of his positions. However, in this case a “haircut” is subtracted from the risky asset's market value to account for the riskiness of the collateral. The haircut is equivalent to a margin since the hedge fund could alternatively have used the risky security to raise cash and then use this cash to cover the margins for asset j . We therefore use the terms margins and haircuts interchangeably.

We have described how funding constraints work when margins and haircuts are set separately for each security position. It is, however, sometimes possible to “cross-margin”, i.e. to jointly finance several trades that are part of the same strategy. This leads to a lower total margin if the risks of the various positions are partially offsetting. For instance, much of the interest rate risk is eliminated in a “spread trade” with a long position in one bond and a short position in a similar bond. Hence, the margin/haircut of a jointly financed spread trade is smaller than the sum of the margins of the long and short bonds. For a strategy that is financed jointly, we can reinterpret security j as such a strategy. Prime brokers compete by, among other things, offering low margins and haircuts — a key consideration for hedge funds — which means that it is becoming increasingly easy to finance more and more strategies jointly. In the extreme, one can imagine a joint financing of a hedge fund's total position such that the “portfolio margin” would be equal to the maximum portfolio loss with a certain confidence level. Currently, it is often not practical to jointly finance a large portfolio. This is because a large hedge fund finances its trades using several brokers, both a hedge fund and a broker can consist of several legal entities (possibly located in different jurisdictions), certain trades need separate margins paid to exchanges (e.g. futures and options) or to other counterparties of the prime broker (e.g. securities lenders), prime brokers may

not have sufficiently sophisticated models to evaluate the diversification benefits (e.g. because they don't have enough data on the historical performance of newer products such as CDOs), and because of other practical difficulties in providing joint financing. Further, if the margin requirement relies on assumed stress scenarios in which the securities are perfectly correlated (e.g. due to predatory trading as in Brunnermeier and Pedersen (2005)), then the portfolio margin constraint coincides with position-by-position margins.

1.2 Funding Requirements for Banks

A bank's capital consists of equity capital plus its long-term borrowings (including credit lines secured from individual or syndicates of commercial banks), reduced by assets that cannot be readily employed (e.g. goodwill, intangible assets, property, equipment, and capital needed for daily operations), and further reduced by uncollateralized loans extended by the bank to others (see e.g. Goldman Sachs 2003 Annual Report). Banks also raise money using short-term uncollateralized loans such as commercial papers and promissory notes, and, in the case of commercial banks, demand deposits. These sources of financing cannot, however, be relied on in times of funding crises since lenders may be unwilling to continue lending, and therefore this short-term funding is often not included in measures of capital.

The financing of a bank's trading activity is largely based on *collateralized* borrowing. Banks can finance long positions using collateralized borrowing from corporations with excess cash, other banks, insurance companies, and the Federal Reserve Bank, and can borrow securities to shortsell from, for instance, mutual funds and pension funds. These transactions typically require margins which must be financed by the bank's capital as captured by the funding constraint (4).

The financing of a bank's proprietary trading is more complicated than that of a hedge fund, however. For instance, banks may negotiate zero margins with certain counterparties, and banks can often sell short shares held in house, that is, held in a customer's margin account (in "street name") such that the bank does not need to use capital to borrow the share externally. Further, a bank receives margins when financing hedge funds (i.e. the margin is negative from the point of view of the bank). However, often the bank wants to pass on the trade to an exchange or another counterparty and hence has to pay a margin to the exchange. In spite of these caveats, we believe that in times of stress, banks face margins and are ultimately subject to a funding constraint in the spirit of (4). For instance, Lehman Brothers, 2001 Annual Report (page 46) states that

“the following must be funded with cash capital: Secured funding ‘hair-cuts,’ to reflect the estimated value of cash that would be advanced to the Company by counterparties against available inventory, Fixed assets and goodwill, [and] Operational cash ... ,”

and Goldman Sachs, 2003 Annual Report (page 62) makes a similar statement.

In addition, banks have to satisfy certain regulatory requirements. Commercial banks are subject to the Basel accord, supervised by the Federal Reserve system for US banks. In short, the Basel accord of 1988 requires that a bank's "eligible capital" exceeds 8% of the "risk-weighted asset holdings," which is the sum of each asset holding multiplied by its risk weight. The risk weight is 0% for cash and government securities, 50% for mortgage-backed loans, and 100% for all other assets. The requirement posed by the 1988 Basel accord corresponds to Equation (4) with margins of 0%, 4%, and 8%, respectively. In 1996, the accord was amended, allowing banks to measure market risk using an internal model based on portfolio VaRs rather than using standardized risk weights.

U.S. broker-speculators, including banks acting as such, are subject to the Securities and Exchange Commission's (SEC's) "net capital rule" (SEC Rule 15c3-1). This rule stipulates, among other things, that a broker must have a minimum "net capital," which is defined as equity capital plus approved subordinate liabilities minus "securities haircuts" and operational charges. The haircuts are set as security-dependent percentages of the market value. The standard rule requires that the net capital exceeds at least $6\frac{2}{3}\%$ (15:1 leverage) of aggregate indebtedness (broker's total money liabilities) or alternatively 2% of aggregate debit items arising from customer transactions. This constraint is similar in spirit to (4).⁷ As of August 20, 2004, SEC amended (SEC Release No. 34-49830) the net capital rule for Consolidated Supervised Entities (CSE) such that CSE's may, under certain circumstances, use their internal risk models to determine whether they fulfill their capital requirement.

1.3 Funding Requirements for Market Makers

There are various types of market-making firms. Some are small partnerships, whereas others are parts of large investment banks. The small firms are financed in a similar way to hedge funds in that they rely primarily on collateralized financing; the funding of banks was described in Section 1.2.

Certain market makers, such as NYSE specialists, have an obligation to make a market and a binding funding constraint means that they cannot fulfill this requirement. Hence, avoiding the funding constraint is especially crucial for such market makers.

Market makers are in principle subject to the SEC's net capital rule (described in Section 1.2), but the rule has special exceptions for market makers. Hence, market makers' main regulatory requirements are those imposed by the exchange on which they operate. These constraints are often similar in spirit to (4).

⁷Let L be the lower of $6\frac{2}{3}\%$ of total indebtedness or 2% of debit items and h^j the haircut for security j ; then the rule requires that $L \leq W - \sum_j h^j x^j$, that is, $\sum_j h^j x^j \leq W - L$.

2 Model

Setup. The economy has J risky assets, traded at times $t = 0, 1, 2, 3$. At time $t = 3$, each security j pays off v^j , a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. There is no aggregate risk and the risk-free interest rate is normalized to zero, so the fundamental value of each stock is its conditional expected value of the final payoff $v_t^j = E_t[v^j]$. Fundamental volatility has an autoregressive conditional heteroskedasticity (ARCH) structure. Specifically, v_t^j evolves according to

$$v_{t+1}^j = v_t^j + \Delta v_{t+1}^j = v_t^j + \sigma_{t+1}^j \varepsilon_{t+1}^j, \quad (1)$$

where all ε_t^j are i.i.d. across time and assets with a standard normal cumulative distribution function Φ with mean zero and unit variance, and the volatility σ_t^j has dynamics

$$\sigma_{t+1}^j = \underline{\sigma}^j + \theta^j |\Delta v_t^j|, \quad (2)$$

where $\underline{\sigma}^j, \theta \geq 0$. A positive θ implies that shocks to fundamentals increase future volatility.

There are three groups of market participants: “customers” and “speculators” trade assets while “financiers” finance speculators’ positions.

There are three risk-averse customers. At time 0, customer $k = 0, 1, 2$ has a cash holding of W_0^k bonds and zero shares, but finds out that he will experience an endowment shock of $\mathbf{z}^k = \{z^{1,k}, \dots, z^{J,k}\}$ shares at time $t = 3$, where \mathbf{z} is a random variables such that the aggregate endowment shock is zero $\sum_{k=0}^2 z^{j,k} = 0$.

The basic liquidity problem arises because customers may arrive sequentially, which gives rise to order imbalance. In particular, customer k only begins trading at time $t = k$ with probability a , while all customers start trading at once at $t = 0$ with probability $(1 - a)$. Before a customer arrives at the market place, his demand is $\mathbf{y}_t^k = 0$. After he arrives, he chooses his security position \mathbf{y}_t^k in order to maximize his exponential utility function $U(W_3^k) = -\exp\{-\rho W_3^k\}$ over final wealth. Wealth W_t^k , including the value of the \mathbf{z}^k shares, evolves according to

$$W_{t+1}^k = W_t^k + (\mathbf{p}_{t+1} - \mathbf{p}_t)' (\mathbf{y}_t^k + \mathbf{z}^k) \quad (3)$$

To total demand shock of customers who have arrived in the market at time t is denoted by $Z_t := \sum_{k=0}^t \mathbf{z}^k$.

The early customers’ trading need is accommodated by speculators who provide liquidity/immediacy. Speculators are risk-neutral and maximize expected final wealth W_3 . Speculators face the constraint that the total margin on their position \mathbf{x}_t cannot

exceed their capital W_t :

$$\sum_j (x_t^{j+} m_t^{j+} + x_t^{j-} m_t^{j-}) \leq W_t \quad (4)$$

where $x_t^{j+} \geq 0$ and $x_t^{j-} \geq 0$ are the positive and negative parts of $x_t^j = x_t^{j+} - x_t^{j-}$, respectively, and $m_t^{j+} \geq 0$ and $m_t^{j-} \geq 0$ are the dollar margin on long and short positions, respectively. Section 1 is devoted to a detailed discussion of the institutional features related to this key constraint.

Speculators start out with an initial cash position of W_0 and zero shares, and their wealth evolves according to

$$W_t = W_{t-1} + (\mathbf{p}_t - \mathbf{p}_{t-1})' \mathbf{x}_{t-1} + \eta_t, \quad (5)$$

where η_t is an independent wealth shock arising from other activities, e.g. the speculators' investment banking arm. When $W_t \leq 0$, speculators go bankrupt and forgo all possible future profit opportunities. We do not grant speculators limited liabilities (i.e. speculators have negative utility for negative wealth). Our results would be qualitatively the same with limited liability.⁸

The financier sets the margins to limit his counterparty credit risk. Specifically, the financier ensures that the margin is large enough to cover the position's π -value-at-risk (where π is a nonnegative number close to zero, e.g. 1%):

$$\pi = Pr(-\Delta p_{t+1}^j > m_t^{j+} | \mathcal{F}_t) \quad (6)$$

$$\pi = Pr(\Delta p_{t+1}^j > m_t^{j-} | \mathcal{F}_t) \quad (7)$$

Equation (6) means that the margin on a long position m^+ is set such that price drops larger than the margin only happen with a small probability π . Similarly, (7) means that price increases larger than the margin on a short position only happen with small probability. Clearly, the margin is larger for more volatile assets. The margin depends on financiers' information set \mathcal{F}_t . We consider two important benchmarks: "informed financiers" who know the fundamental value and the liquidity shocks z , $\mathcal{F}_t = \sigma\{\mathbf{z}, \mathbf{v}_0, \dots, \mathbf{v}_t, \mathbf{p}_0, \dots, \mathbf{p}_t, \eta_1, \dots, \eta_t\}$, and "uninformed financiers" who only observes prices, $\mathcal{F}_t = \sigma\{\mathbf{p}_0, \dots, \mathbf{p}_t\}$. This margin specification is motivated by the real-world institutional features described in Section 1. Theoretically, Stiglitz and Weiss (1981) show how credit rationing can be due to adverse selection and moral hazard in the lending market, and Geanakoplos (2003) considers endogenous contracts in a general-equilibrium framework of imperfect commitment.

⁸We could allow the speculators to raise new capital as long as this takes time. Indeed, the model would be the same if the speculators could raise capital only at time 2 (and in this case we need not assume that the customers' endowment shocks z^j aggregate to zero).

We let Λ_t^j be the (signed) deviation of the price from fundamental value

$$\Lambda_t^j = p_t^j - v_t^j \quad (8)$$

and we define our measure of market illiquidity as the absolute amount of this deviation, $|\Lambda_t^j|$. We consider competitive equilibria of the economy:

Definition 1 *An equilibrium is a price process \mathbf{p}_t such that (i) \mathbf{x}_t maximizes the speculators' expected final profit subject to the margin constraint (4), (ii) each \mathbf{y}_t^k maximizes k -customers expected utility after their arrival at the marketplace and is zero beforehand, (iii) margins are set according to the VaR specification (6), and (iv) markets clear, $\mathbf{x}_t + \sum_{k=0}^2 \mathbf{y}_t^k = 0$.*

Outline of Equilibrium. We derive the optimal strategies for customers and speculators using dynamic programming, starting from time 2, and working backwards. The customers' value function is denoted Γ and the speculators' is denoted J . At time 2, customer k 's problem is

$$\Gamma_2(W_2^k, p_2, v_2) = \max_{y_2^k} -E_2[e^{-\gamma W_3^k}] \quad (9)$$

$$= \max_{y_2^k} -e^{-\gamma(E_2[W_3^k] - \frac{\gamma}{2} \text{Var}_2[W_3^k])} \quad (10)$$

which has the solution

$$y_2^{j,k} = \frac{v_2^j - p_2^j}{\gamma(\sigma_3^j)^2} - z^{j,k} \quad (11)$$

Clearly, since all customers are present in the market at time 2, the unique equilibrium is $p_2 = v_2$. Indeed, when the prices are equal to fundamentals the aggregate customer demand is zero, $\sum_k y_2^{j,k} = 0$, and the speculator also has a zero demand. We get the customer's value function $\Gamma_2(W_2^k, p_2 = v_2, v_2) = -e^{-\gamma W_2^k}$, and the speculator's value function $J_2(W_2, p_2 = v_2, v_2) = W_2$.

The equilibrium before time 2 depends on whether the customers arrive sequentially or all at time 0. If all customers arrive at time 0, then the simple arguments above show that $p_t = v_t$ at any time $t = 0, 1, 2$.

We are interested in the case with sequential arrival of the customers such that the speculators' liquidity provision is needed. At time 1, customers 0 and 1 are present in the market, but customer 2 has not arrived yet. As above, customer $k = 0, 1$ has a

demand and value function of

$$y_1^{j,k} = \frac{v_1^j - p_1^j}{\gamma(\sigma_2^j)^2} - z^{j,k} \quad (12)$$

$$\Gamma_1(W_1^k, p_1, v_1) = -\exp \left\{ -\gamma \left[W_1^k + \sum_j \frac{(v_1^j - p_1^j)^2}{2\gamma(\sigma_2^j)^2} \right] \right\} \quad (13)$$

At time 0, customer $k = 0$ arrives in the market and maximizes $E_0[\Gamma_1(W_1^k, p_1, v_1)]$.

At time $t = 1$, if the market is perfectly liquid so that $p_1^j = v_1^j$ for all j , then the speculator is indifferent between all possible positions x_1 . If some securities have $p_1 \neq v_1$, then the risk neutral speculator invests all his capital such that his margin constraint binds. The speculator optimally trades only in securities with the highest expected profit per dollar used. The profit per dollar used is $(v_1^j - p_1^j)/m_1^{j+}$ on a long position and $-(v_1^j - p_1^j)/m_1^{j-}$ on a short position. A speculator's shadow cost of capital, denoted ϕ_1 , is 1 plus the maximum profit per dollar used as long as he is not bankrupt:

$$\phi_1 = 1 + \max_j \left(\max \left\{ \frac{v_1^j - p_1^j}{m_1^{j+}}, \frac{-(v_1^j - p_1^j)}{m_1^{j-}} \right\} \right) \quad (14)$$

where the margins for long and short positions are set by the financier as described in the next section. If the speculator is bankrupt $W_1 < 0$ then $\phi_1 = 1$. Each speculator's value function is therefore

$$J_1(W_1, p_1, v_1, p_0, v_0) = W_1 \phi_1 \quad (15)$$

At time $t = 0$, the speculator maximizes $E_0[W_1 \phi_1]$ subject to his capital constraint (4).

The equilibrium prices at times 1 and 0 do not have simple expressions. However, after having derived the margin conditions, we characterize several important properties of these prices, which illuminates the connection between market liquidity, $|\Lambda|$, and speculators' funding situation.

3 Margin Setting and Liquidity (Time 1)

We want to determine the financiers' margin, m_1 , at time 1, both in the case informed and uninformed financiers. The financier sets the margin such that it covers the position's value-at-risk, knowing that prices equal the fundamental in the next period, $p_2 = v_2$.

If informed financiers know the fundamental values v_1 (or, equivalently, know the demand shocks z_0, z_1), they are aware of Λ_1 . Since $\Lambda_2 = 0$, margins on long positions

at $t = 1$ are set according to

$$\begin{aligned}
\pi &= Pr(-\Delta p_2^j > m_1^{j+} | \mathcal{F}_1) \\
&= Pr(-\Delta v_2^j + \Lambda_1^j > m_1^{j+} | \mathcal{F}_1) \\
&= 1 - \Phi\left(\frac{m_1^{j+} - \Lambda_1^j}{\sigma_2^j}\right)
\end{aligned} \tag{16}$$

which implies that

$$\begin{aligned}
m_1^{j+} &= \Phi^{-1}(1 - \pi) \sigma_2^j + \Lambda_1^j \\
&= \bar{\sigma}^j + \bar{\theta} |\Delta v_1^j| + \Lambda_1^j
\end{aligned} \tag{17}$$

where we define

$$\bar{\sigma}^j = \underline{\sigma}^j \Phi^{-1}(1 - \pi) \tag{18}$$

$$\bar{\theta} = \theta \Phi^{-1}(1 - \pi) \tag{19}$$

The margin on a short position can be derived similarly:

Proposition 1 (Stabilizing Margins and the Cushioning Effect) *When the financier is informed about the fundamental value and knows that prices will equal fundamentals next period $t = 2$, then the margins on long and short positions are, respectively,*

$$m_1^{j+} = \max\{\bar{\sigma}^j + \bar{\theta} |\Delta v_1^j| + \Lambda_1^j, 0\} \tag{20}$$

$$m_1^{j-} = \max\{\bar{\sigma}^j + \bar{\theta} |\Delta v_1^j| - \Lambda_1^j, 0\} \tag{21}$$

The more prices are below fundamentals $\Lambda_1^j < 0$, the lower is the margin on a long position m_1^{j+} , and the more prices are above fundamentals $\Lambda_1^j > 0$, the lower is the margin on a short position m_1^{j-} . Hence, in this case illiquidity reduces margins for speculators who buy low and sell high.

The margins are reduced by illiquidity because the speculator is expected to profit when prices return to fundamentals at time 2, and this profit ‘‘cushions’’ the speculators from losses due to fundamental volatility. Thus, we denote the margins set by informed financiers at $t = 1$ as *stabilizing margins*.

Stabilizing margins are an interesting benchmark, and they are hard to escape in a theoretical model. However, real-world liquidity crisis are often associated with increases in margins, not decreases. To capture this, we turn to the case of a financier who is uninformed about the current fundamental so that he must set his margin based on the observed prices p_0 and p_1 . This is in general a complicated problem since the financier needs to filter out the probability that a liquidity shock occurred and the

values of z_0 and z_1 . The expression becomes simple, however, if the financiers' prior probability of a liquidity shock is small so that he finds it likely that $p_t^j = v_t^j$, implying a common margin $m_1^j = m_1^{j+} = m_1^{j-}$ for long and short positions in the limit:

Proposition 2 (Destabilizing Margins) *When the financier is uninformed about the fundamental value, then as $a \rightarrow 0$ the margins on long and short positions approach*

$$m_1^j = \bar{\sigma}^j + \bar{\theta}|\Delta p_1^j| = \bar{\sigma}^j + \bar{\theta}|\Delta v_1^j + \Delta\Lambda_1^j| \quad (22)$$

Margins are increasing in price volatility and illiquidity shocks can increase margins.

Intuitively, since liquidity risk tends to increase the price volatility, and since an uninformed financier may interpret price volatility as fundamental volatility, this increases margins.⁹ Equation (22) corresponds closely to the margin setting in reality which is primarily based on volatility estimates using past price movements. Equation (22) shows that illiquidity increases margins when the liquidity shock $\Delta\Lambda_1^j$ has the same sign as the fundamental shock Δv_1^j or is greater in magnitude, but margins are reduced if the liquidity shocks counterbalances a fundamental move. We denote the phenomenon that margins can increase as illiquidity rises by *destabilizing margins*. As we will see next, the information of the financier — i.e. whether margins are stabilizing or destabilizing — has important implications for the equilibrium.

4 Fragility and Liquidity Spirals (Time 1)

Fragility. This section discusses liquidity spirals and fragility — the property that a small change in fundamentals can lead to a large jump in liquidity. For simplicity we illustrate this with a single security $J = 1$.

We say that liquidity is fragile if the equilibrium price $p_t(\eta_t, v_t)$ cannot be chosen to be continuous in the exogenous shocks, namely η_t and Δv_t . Fragility arises when the excess demand for shares $x_t + \sum_{k=0}^1 \mathbf{y}_1$ can be non-monotonic in the price. While under “normal” circumstances a high price leads to a low total demand – excess demand is decreasing – binding funding constraints along with destabilizing margins (margin effect) or speculators' losses (loss effect) can lead to an increasing demand curve.

It is natural to focus on *stable* equilibria. An equilibrium is stable if a small negative (positive) price perturbation leads to excess demand (supply), which, intuitively, “pushes” the price back up (down) to its equilibrium level.

Proposition 3 (Fragility) *(i) With informed financiers, the market is fragile at time 1 if x_0 is large enough.*

⁹In the analysis of time 0, we shall see that margins can also be destabilizing when price volatility signals future liquidity risk (not necessarily fundamental risk).

(ii) *With uninformed financiers, the market is fragile at time 1 if x_0 large enough or if the ARCH parameter, θ , is large enough and the financiers' prior on customer shock z -shock, a , is small enough.*

Numerical Example. We illustrate how fragility arises due to destabilizing margins or dealer losses by way of a numerical example. We consider the more interesting (and arguably more realistic) case in which the financiers are uninformed and choose parameters as follows.

The fundamental value has ARCH volatility parameters $\underline{\sigma} = 5$ and $\theta = 0.3$, which implies clustering of volatility. The price in the previous time period was $p_0 = 130$, the aggregate demand shock of the customers who have arrived at time 1 is $Z_1 = z_0 + z_1 = 40$, and the customers' risk aversion is $\gamma = 0.025$. The speculators' have an initial position of $x_0 = 0$ and a cash wealth of $W_1 = 750$. Finally, the financier uses a VaR with $\pi = 1\%$ and $a \rightarrow 0$, leading to a margin requirement of $m = \bar{\sigma} + \bar{\theta}|\Delta p_1| = 2.326(5 + 0.3|\Delta p_1|)$.

We first show how the speculators' choose their optimal time-1 position x_1 , and how the equilibrium price p_1 depends on the exogenous shocks to their wealth η_1 and to the security's fundamentals Δv_1 . Panel A of Figure 2 illustrates how the speculators' demand x_1 and the customers' supply (i.e. the negative of the customers' demand as per Equation (12)) depend on the price p_1 when the fundamental value is $v_1 = 120 < p_0 = 130$ and the speculators' wealth shock is $\eta_1 = 0$. Customers' supply is given by the upward sloping line since, naturally, their supply is greater if the price is greater. Customers supply $Z_1 = 40$ shares, namely the shares that they anticipate receiving at time $t = 3$, when the market is perfectly liquid $p_1 = v_1$ (i.e. illiquidity is $|\Lambda_1| = 0$). For lower prices they throw fewer shares on the market.

The speculators must satisfy their margin constraints $|x_1| \leq W_1/(\bar{\sigma} + \bar{\theta}|\Delta p_1|)$, which graphically means that their demand must be inside the "hyperbolic star" defined by the four (dashed) hyperbolas. At the price $p_1 = p_0 = 130$, the margin is smallest and hence the constraint is most relaxed. As p_1 departs from $p_0 = 130$, margins increase and speculators become more constrained — the horizontal distance between two hyperbola shrinks.

The speculators' demand curve is given by the solid line in the figure: For $p_1 = v_1 = 120$, the security's expected return is zero and the speculator is indifferent between all his possible positions on the horizontal line. For price levels $p_1 < v_1$ below this line, the risk neutral speculators want to buy the asset, $x_1 > 0$, and their demand is constrained by the lower right side of the star. Similarly, for prices above v_1 , speculators short the asset, $x_1 < 0$, and their demand is limited by the left side of the star. Interestingly, the speculators' demand is upward sloping for prices below $v_1 = 120$ because a larger price drop Δp_1 increases the financiers' estimate of fundamental volatility and consequently margins.

There are two stable equilibria: a perfect liquidity equilibrium with price $p_1 =$

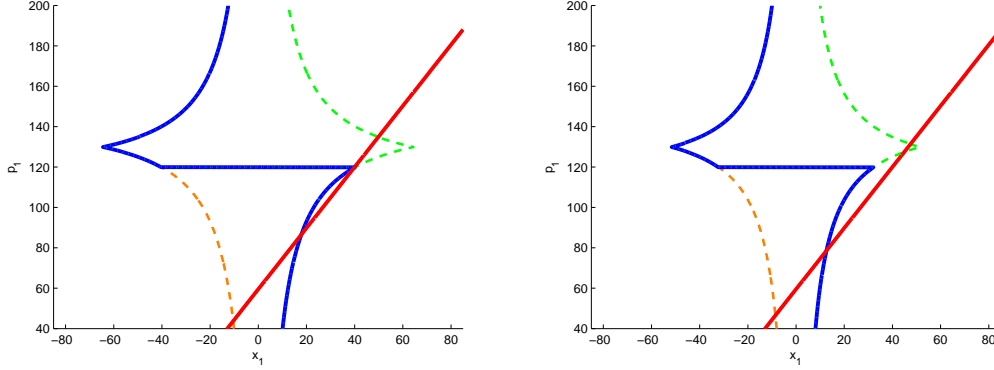


Figure 2: **Speculator Demand and Customer Supply.** The financiers are uninformed and the fundamentals have volatility clustering leading to destabilizing margins. Specifically, the speculators’ margin constraint means that their position must be inside the four dashed lines (the “hyperbolic star”). The solid curve is the speculators’ optimal demand. The upward sloping line is the customers’ supply, that is, the negative of their demand. In Panel A the speculators experienced no wealth shock, $\eta_1 = 0$, while in Panel B they faced a negative wealth shock of $\eta_1 = -150$, otherwise everything is the same. In Panel A, perfect liquidity $p_1 = v_1 = 120$ is one of two stable equilibria, while in Panel B the unique equilibrium is illiquid.

$v_1 = 120$ and an illiquid equilibrium with a price of about 86.5 (and an uninteresting unstable equilibrium in between the two stable ones).

Panel B of Figure 2 shows the same plot as Panel A, but with a negative wealth shock to speculators of $\eta_1 = -150$ instead of $\eta_1 = 0$. In this case, perfect liquidity with $p_1 = v_1$ is no longer an equilibrium since the speculator cannot fund a large enough position. The unique equilibrium is highly illiquid because of the speculators’ lower wealth and, importantly, because of the endogenously higher margins.

This “disconnect” between the perfect-liquidity equilibrium and the illiquid equilibrium and the resulting fragility is more directly illustrated in Figure 3. Panel A plots the equilibrium price correspondence for different exogenous funding shocks η_1 (with fixed $\Delta v_1 = -5$) and shows that a marginal reduction in funding cannot always lead to a smooth reduction in market liquidity. Rather, there must be a level of funding such that an infinitesimal drop in funding leads to a discontinuous drop in market liquidity.

Panel B of Figure 3 plots the equilibrium price correspondence for different realizations of the fundamental shock Δv_1 (with fixed $\eta_1 = 0$) and shows the same form of discontinuity for adverse fundamental shocks v_1 . The discontinuity in Δv_1 is most easily understood in conjunction of Figure 2 Panel A. As Δv_1 falls, the horizontal line of speculators’ demand shifts downwards and the perfect liquidity equilibrium vanishes.

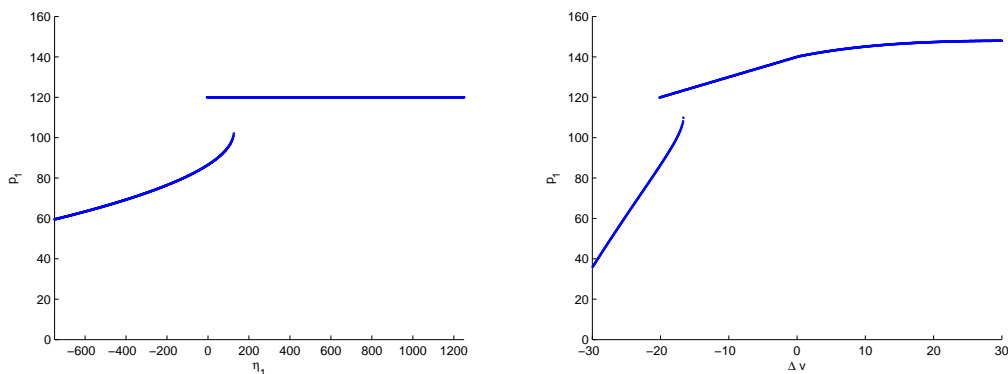


Figure 3: **Fragility due to Destabilizing Margins.** The figure shows the equilibrium price correspondence as a function of the speculators' wealth shock η_1 (Panel A) and of fundamental shocks Δv_1 (Panel B). The parameters are as in Figure 2. The margins are destabilizing since financiers are uninformed and fundamentals exhibit volatility clustering. The discontinuity of the equilibrium prices reflects fragility in liquidity since a small shock can lead to a disproportionately large price effect.

Panel B also reveals an interesting asymmetry that negative fundamental shocks leads to larger price movements than corresponding positive shocks (for $Z_1 > 0$).

Fragility can also arise because of shocks to customer demand or volatility. For instance, the price correspondence p_1 is discontinuous in the customers' shock Z_1 since a larger Z_1 shifts customers' supply in Figure 2 down such that the perfect liquidity equilibrium vanishes. When this happens, a marginally larger demand for liquidity by customers leads, paradoxically, to a drastic reduction of liquidity supply by the speculators.

So far we have considered speculators with zero initial positions, $x_0 = 0$. If $x_0 > 0$, then lower prices lead to losses for the speculators, and graphically this means that the constraints in the “hyperbolic star” tighten (i.e. the gap between the hyperbolas narrows) for low prices. Because of this “loss effect,” the discontinuous price drop associated with the illiquid equilibrium is even larger.

The discontinuity in prices driven by destabilizing margins and dealer losses can help to explain the sudden market liquidity dry-ups observed in many markets. For example, Russia's default in 1998 was in itself only a trivial wealth shock relative to global arbitrage capital. Nevertheless, it had a large effect on liquidity in global financial markets, consistent with our fragility result that a small wealth shock can push the equilibrium over the edge.

Liquidity Spirals. Once the economy enters into the illiquid equilibrium, market liquidity becomes highly sensitive to shocks. We identify two amplification mechanisms, a “margin spiral” due to increasing margins as speculator financing worsens, and a “loss spiral” due to escalating speculator losses. Figure 4 illustrates these “liquidity

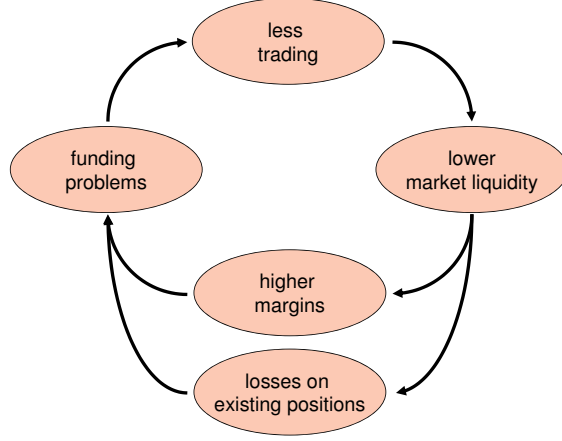


Figure 4: **Liquidity Spirals**

spirals”: A shock to speculators’ capital, $\eta_1 < 0$, forces speculators to provide less market liquidity, which increases the price impact of the customers’ demand pressure. With uninformed financiers and a positive ARCH effect $\theta > 0$, the resulting price swing increases financiers’ estimate of the fundamental volatility and, hence, increases the margin, thereby worsening the speculators’ funding problems even further, and so on, leading to a “margin spiral.” Similarly, increased market illiquidity can lead to losses on the speculators’ existing positions, worsening their funding problem and so on, leading to a loss spiral. Mathematically, the spirals are seen as follows:

Proposition 4

- (i) *If speculators’ capital constraint is slack then the price p_1 is equal to v_1 and insensitive to local changes in speculator wealth.*
- (ii) **(Liquidity Spirals)** *In a stable illiquid equilibrium with selling pressure from customers, $Z_1, x_1 > 0$, the price sensitivity to speculator wealth shocks η_1 is*

$$\frac{\partial p_1}{\partial \eta_1} = \frac{1}{\frac{2}{\gamma(\sigma_2)^2} m_1^+ + \frac{\partial m_1^+}{\partial p_1} x_1 - x_0} \tag{23}$$

and with buying pressure from customers, $Z_1, x_1 < 0$,

$$\frac{\partial p_1}{\partial \eta_1} = \frac{-1}{\frac{2}{\gamma(\sigma_2)^2} m_1^- + \frac{\partial m_1^-}{\partial p_1} x_1 + x_0}. \tag{24}$$

A margin spiral arises if $\frac{\partial m_1^+}{\partial p_1} < 0$ or $\frac{\partial m_1^-}{\partial p_1} > 0$, which happens with positive probability if speculators are uninformed and a is small enough. A loss spiral arises if speculators' previous position is in the opposite direction as the demand pressure $x_0 Z_1 > 0$.

This proposition is intuitive. Imagine first what happens if the speculators face a wealth shock of \$1, margins are constant, and speculator has no inventory $x_0 = 0$. In this case, the speculator must reduce his position by $1/m_1$. Since the slope of each of the two customers' demand curve is¹⁰ $1/(\gamma(\sigma_2)^2)$, we get a total price effect of $1/(\frac{2}{\gamma(\sigma_2)^2}m_1)$.

The two additional terms in the denominator imply amplification or dampening effects due to changes in the margin requirement and to profit/losses on the speculators' existing positions. To see that, recall that for any $k > 0$ and l with $|l| < k$ it holds that $\frac{1}{k-l} = \frac{1}{k} + \frac{l}{k^2} + \frac{l^2}{k^3} + \dots$ so with $k = \frac{2}{\gamma(\sigma_2)^2}m_1$ and $l = \frac{\partial m_1^\pm}{\partial p_1}x_1 \pm x_0$ such that each term in this infinite series corresponds to one loop around the circle in Figure 4. The total effect of the changing margin and the speculators' position amplifies the effect if $l > 0$. Intuitively, if e.g. $Z_1 > 0$ then customers' selling pressure is pushing down the price, and $\frac{\partial m_1^+}{\partial p_1} < 0$ means that as prices go down, margins increase, making speculators' funding tighter and thus destabilizing the system. Similarly, when customers are buying, $\frac{\partial m_1^-}{\partial p_1} > 0$ implies that increasing prices leads to increased margins, making it harder for speculators to shortsell, thus destabilizing the system. The system is also destabilized if speculators' losses money on their previous position as prices move away from fundamentals similar to e.g. Shleifer and Vishny (1997).

Interestingly, the total effect of a margin spiral and a loss spiral is greater than the sum of their separate effects. This can be seen mathematically by using simple convexity arguments, and it can be seen intuitively from the flow diagram of Figure 4.

Note that spirals can also be "started" by a shock to liquidity demand Z_1 , fundamentals v_1 , or volatility. It is straightforward to compute the price sensitivity with respect to such shocks. They are just multiples of $\frac{\partial p_1}{\partial \eta_1}$. For instance, a fundamental shock affects the price both because of its direct effect on the value and because of its effect on the customer's estimate of future volatility — and both of these effects are amplified by the liquidity spirals.

Our analysis sheds some new light on the 1987 stock market crash, complementing the standard culprit, portfolio insurance trading. In the 1987 stock market crash numerous market makers hit (or violated) their funding constraint:

"By the end of trading on October 19, [1987] thirteen [NYSE specialist] units had no buying power"
— SEC (1988), page 4-58

Several of these firms managed to reduce their positions and continue their operations. Others did not. For instance, Tompane was so illiquid that it was taken over by

¹⁰See Equation (12).

Merrill Lynch Specialists and Beauchamp was taken over by Spear, Leeds & Kellogg (Beauchamp's clearing broker). Also, market makers outside the NYSE experienced funding troubles: the Amex market makers Damm Frank and Santangelo were taken over; at least 12 OTC market makers ceased operations; and several trading firms went bankrupt.

These funding problems were due to (i) reductions in capital arising from trading losses and default on unsecured customer debt, (ii) an increased funding need stemming from increased inventory and, (iii) increased margins. One New York City bank, for instance, increased margins/haircuts from 20% to 25% for certain borrowers, and another bank increased margins from 25% to 30% for all specialists (SEC (1988) page 5-27 and 5-28). Other banks reduced the funding period by making intra-day margin calls, and at least two banks made intra-day margin calls based on assumed 15% and 25% losses, thus effectively increasing the haircut by 15% and 25%. Also, some broker-speculators experienced a reduction in their line of credit and – as Figure 1 shows – margins at the futures exchanges also drastically increased. (See SEC (1988) and Wigmore (1998).)

In summary, our results on fragility and liquidity spirals imply that during “bad” times, small changes in underlying funding conditions (or liquidity demand) can lead to sharp reductions in liquidity. The 1987 crash exhibited several of the predicted features, namely capital constrained dealers, increased margins, and increased illiquidity.

5 Commonality and Flight to Quality (Time 1)

We now turn to the cross-sectional implications of illiquidity. Since speculators are risk-neutral, they optimally invest all their capital in securities that have the greatest expected profit $|\Lambda^j|$ per capital use, i.e. per dollar margin m^j , as expressed in Equation (14). That equation also introduces the shadow cost of capital ϕ_1 as the marginal value of an extra dollar. The speculators' shadow cost of capital ϕ_1 captures well the notion of funding liquidity. Indeed, a high ϕ means that the available funding — from capital W_1 and from collateralized financing with margins m_1^j — is low relative to the needed funding, which depends on the investment opportunities deriving from demand shocks z^j .

The market liquidity of all assets depend on the speculators' funding liquidity, especially high-margin assets, and this has several interesting implications:

Proposition 5 *Suppose θ^j close enough to zero for all j , and either financiers are informed or uninformed with probability a small enough. Then we have:*

(i) (Commonality of Market Liquidity) *The market illiquidity $|\Lambda|$ of any two securities k and l comove,*

$$\text{Cov}_0 (|\Lambda_1^k|, |\Lambda_1^l|) \geq 0 \tag{25}$$

and market illiquidity comoves with funding illiquidity as measured by the speculators' shadow cost of capital ϕ_1

$$\text{Cov}_0 [|\Lambda_1^k|, \phi_1] \geq 0 \quad (26)$$

(ii) (Commonality of Fragility) *Jumps in market liquidity occurs simultaneously for all assets for which speculators are marginal investors.*

(iii) (Quality=Liquidity) *If asset l has lower fundamental volatility than asset k , $\underline{\sigma}^l < \underline{\sigma}^k$, then l also has lower market illiquidity,*

$$|\Lambda_1^l| \leq |\Lambda_1^k| \quad (27)$$

(iv) (Flight to Quality) *The market liquidity differential between high- and low-fundamental-volatility securities is bigger when speculator funding is tight, that is, $\underline{\sigma}^l < \underline{\sigma}^k$ implies that $|\Lambda_1^k|$ increases more with a negative wealth shock to the speculator,*

$$\frac{\partial |\Lambda_1^l|}{\partial (-\eta_1)} \leq \frac{\partial |\Lambda_1^k|}{\partial (-\eta_1)} \quad (28)$$

if $x_1^k \neq 0$. Further, if with large enough probability $x^k \neq 0$, then

$$\text{Cov}_0(|\Lambda_1^l|, \phi_1) \leq \text{Cov}_0(|\Lambda_1^k|, \phi_1). \quad (29)$$

Numerical Example, Continued. To illustrate these cross-sectional predictions, we extend the numerical example of Section 4 to having two securities. The two securities only differ in their long-run fundamental volatility: $\bar{\sigma}^1 = 5$ and $\bar{\sigma}^2 = 7.5$. The other parameters are as before, except that we double W_1 to 1500 since the speculators now trade two securities, and the financiers remain uninformed.

Figure 5 depicts the assets' equilibrium price for different values of the funding shock η_1 . First note that as speculators' funding tightens and our funding illiquidity measure ϕ_1 rises, the market illiquidity measure $|\Lambda_1^j|$ rises for both assets. Hence, for random η_1 $\text{Cov}_0 [|\Lambda_1^k|, |\Lambda_1^l|] > 0$, our commonality in liquidity result.

“Commonality in fragility” cannot directly be seen from Figure 5, but the fact that the range of η_1 for which there are multiple prices p_1^j coincides for both assets is indicative of it. The intuition for this result is the following. Whenever funding is unconstrained then there is perfect market liquidity provision for all assets. If funding is constrained, then it cannot be the case that speculators provide perfect liquidity for one asset but not for the other one, since they always would have an incentive to shift funds towards the asset with non-perfect market liquidity. Hence, market illiquidity jumps for both assets exactly at the same funding level.

Our result that relates fundamental volatility to market liquidity (“Quality=Liquidity”) is reflected in that p_1^2 is below p_1^1 for any given funding level. Hence, the high-fundamental-volatility asset 2 is always less liquid than the low-fundamental-volatility

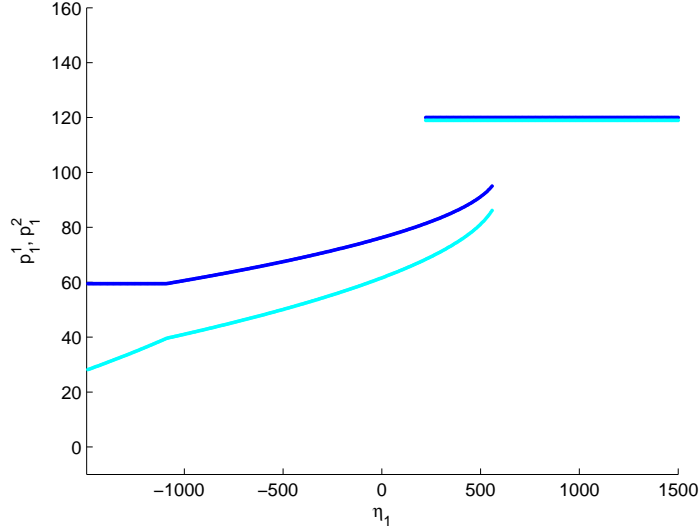


Figure 5: **Flight to Quality and Commonality in Liquidity.** The figure plots that price p_1^j of assets 1 and 2 as functions of speculators' funding W_0 . Asset 1 has lower long-run fundamental risk than asset 2, $\bar{\sigma}^1 = 5 < 7.5 = \bar{\sigma}^2$.

asset 1.

The graph also illustrates our result on “Flight to Quality” To see this, consider the two securities' relative price sensitivity with respect to η_1 . For large wealth shocks, market liquidity is perfect for both assets, i.e. $p_1^1 = p_1^2 = v_1^1 = v_1^2 = 120$ so in this high range of funding, market liquidity is insensitive to marginal changes in funding. For sufficiently small funding levels, $\eta_1 < 222$, market illiquidity of both assets increases since speculators must take smaller stakes in both asset. Importantly, as funding decreases, p_1^2 decreases more steeply than p_1^1 , that is, asset 2 is more sensitive to funding declines than asset 1. This is because speculators cut back more on the “funding-intensive” asset 2 with high margin requirement. The speculators want to maximize their profit per dollar margin, $|\Lambda^j|/m^j$ and therefore $|\Lambda^2|$ must be higher than $|\Lambda^1|$ to compensate speculators for using more capital for margin.

For sufficiently low funding levels, speculators put all their funds into asset 2. Since asset 2 is more volatile, risk averse customers are more eager to sell this stock. Hence, the price is lower and it is more attractive to provide market liquidity for asset 2 for very low levels of funding.

6 Liquidity Risk (Time 0)

In this section, we focus on $t = 0$ and demonstrate that (i) funding liquidity risk matters even before margin requirements actually bind, (ii) the pricing kernel depends on future funding liquidity, ϕ_{t+1} , (iii) the conditional distribution of prices p_1 is skewed due to the funding constraint, and (iv) margins m_0 and illiquidity Λ_0 can be positively related due to liquidity risk even if financiers are informed.

Speculators trade cautiously if they face a risk of binding margin constraints over the life of the trade. To see this, we consider the speculators' trading decision at time 0, which reflects his time-1 liquidity risk. He maximizes $E_0[\phi_1 W_1]$ subject to his margin constraint. The wealth is linear in the position x_0 , but x_0 also affects ϕ_1 since a larger $|x_0|$ expands the bankruptcy region, and this latter effect in fact makes the problem concave. If the speculator is not constrained at time $t = 0$, then the first-order condition for his position in security j is $E_0[\phi_1(p_1^j - p_0^j)] = 0$. (We leave the details of these calculations for the appendix, including the case of a constrained time-0 speculator.) This implies that the funding liquidity ϕ_1 is the pricing kernel for the cross section of securities:

$$p_0^j = \frac{E_0[\phi_1 p_1^j]}{E_0[\phi_1]} = E_0[p_1^j] + \frac{Cov_0[\phi_1, p_1^j]}{E_0[\phi_1]}. \quad (30)$$

This shows that the price at time 0 is the expected time-1 price — which already depends on the liquidity issues at time-1 — further adjusted for liquidity risk in the form of a covariance term. The liquidity risk term is intuitive: The time-0 price is lower if the covariance is negative, that is, if the security has a low payoff during future funding liquidity crisis when ϕ_1 is high.

These type of considerations appear to play a role in financial institutions' risk management. They keep some cash on the side-line to ensure that they can exploit potential future investment opportunities.¹¹ For instance, Goldman Sachs (2003 Annual Report, page 62) states that it seeks to maintain net capital in excess of total margins and haircuts that it would face in periods of market stress plus the total draws on unfunded commitments at such times. In addition, Goldman Sachs recognizes that it may not have access to short-term borrowing during a crisis, that margins and haircuts may increase during such a crisis, and that counterparties may withdraw funds at such times.

¹¹Note that an adverse shock lowers speculators' wealth at $t = 1$, but creates a profitable investment opportunity in $t = 1$. One might think that the latter effect provides a natural "dynamic hedge" and hence, speculators (with a relative risk aversion coefficient larger than one) increase their $t = 0$ -hedging demand, which in turn, lowers illiquidity in $t = 0$. However, exactly the opposite occurs in a setting with capital constraints. Capital constraints prevent speculators from taking advantage from this investment opportunities in $t = 1$. Hence, speculators are reluctant to trade away the illiquidity in $t = 0$. In this sense, our mechanism is different from one that is driven by risk-aversion.

The events following the Russian default in 1998 is another stark example of funding liquidity risk. The hedge fund Long Term Capital Management (LTCM) had been aware of funding liquidity risk. Indeed, they estimated that in times of severe stress, haircuts on AAA-rated commercial mortgages would increase from 2% to 10%, and similarly for other securities (HBS Case N9-200-007(A)). In response to this, LTCM had negotiated long-term financing with margins fixed for several weeks on many of their collateralized loans. Other firms with similar strategies, however, experienced increased margins. Due to an escalating liquidity spiral, LTCM could ultimately not fund its positions in spite of their numerous measures to control funding risk and was taken over by 14 banks in September 1998.

Numerical Example. To better understand funding liquidity risk, we explore the time-0 properties of our model using a numerical example with one security. We let $\gamma = 0.005$, $\underline{\sigma} = 5$, $z_0 = 40$, $z_1 = 0$, $v_0 = 140$, $E[v_1] = 140$, $\theta = 0.1$, $\pi = 0.5\%$, and $\eta_1 = 0$. We consider first an uninformed financier and later turn to an informed one. We do so in order to avoid excessive risk-taking for very low values of W_0 .

Figure 6 depicts the price p_0 and expected time-1 price $E_0(p_1)$ for different initial wealth levels, W_0 , for which the speculators' funding constraint is not binding in $t = 0$. The figure shows that even though the speculators' are unconstrained at time 0, market liquidity provision is limited with prices below the fundamental value of $E_0[v] = 140$. The price is below the fundamental for two reasons: First, the expected time-1 price is below the fundamental value because of the risk that speculators cannot accommodate the customers' selling pressure at that time. Second, holding the security leads to losses in the states of nature when speculators are constrained and investment opportunities are good, which makes speculators require an additional compensation for holding the asset. Interestingly, for very low wealth levels, the second effect is reversed. This can happen in part because large positive fundamental shocks to the asset leads to high volatility estimates, increasing margins, and thus making dealers constrained and illiquidity large, and this gives the speculators an incentive to hold the security.

Another interesting feature of the model is that the funding constraint introduces skewness in the p_1 -distribution conditional on the sign of the demand pressure. For $Z_1 > 0$ speculators take long positions and, consequently, large negative v_1 -shocks lead to capital losses. This loss-effect triggers a sharper price reaction than corresponding positive v_1 -shocks because of the liquidity spirals. Figure 7 shows this negative skewness which is especially pronounced for low wealth levels (although the effect is not monotone — zero dealer wealth implies no skewness, for instance).

Of course, for correspondingly negative realizations of Z_1 , funding constraints induce a positive skewness in the p_1 -distribution — but still negative skewness for the speculators' return. This suggests from an ex-ante view, prior to the realization of Z_1 , that funding constraints lead to higher kurtosis of the price distribution.

Finally, unlike at time $t = 1$, margins can be positively related to illiquidity at

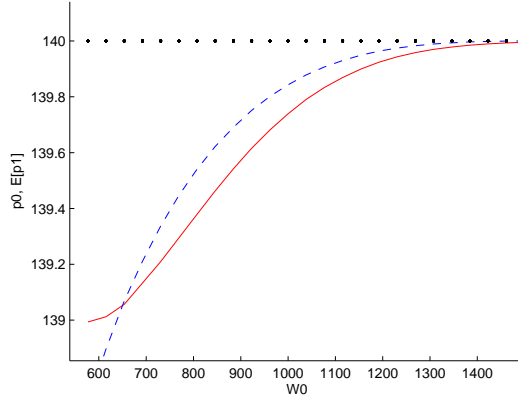


Figure 6: **Illiquidity at Time 0.** The solid line plots p_0 for different funding levels W_0 . The dashed line depicts $E_0[p_1]$, while the dotted horizontal line shows $v_0 = 140$. The difference between p_0 and $v_0 = 140$ reflects illiquidity. It arises even though the constraint is not binding for all depicted W_0 .

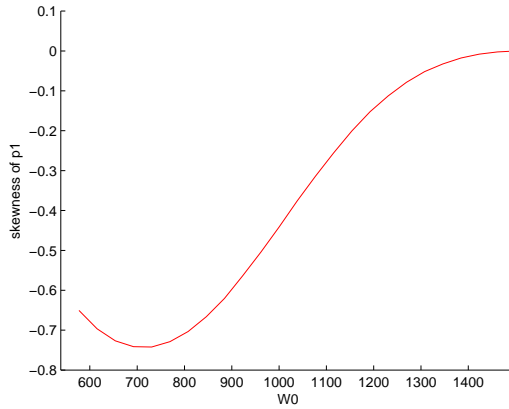


Figure 7: **Skewness of p_1 .** The figure depicts the skewness for different funding levels W_0 for which the funding constraint is not binding.

time 0, even when the financiers' are fully informed. This is because of the liquidity risk between time 0 and time 1. To see this note that if we reduce the speculators initial wealth W_0 , then the market becomes less liquid in the sense that the price is further from the fundamental value. At the same time, the equilibrium price in $t = 1$ is more volatile and thus equilibrium margin at time zero actually increases as shown

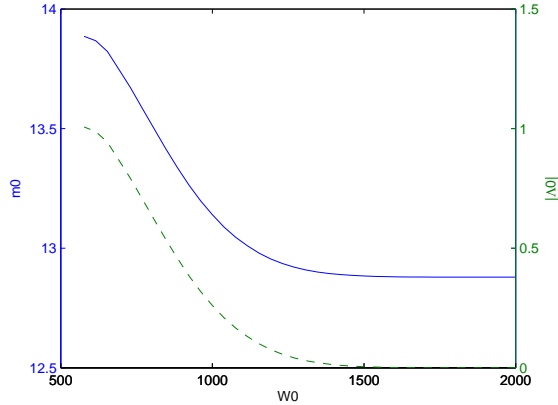


Figure 8: **Margins and Illiquidity at Time 0.** The solid line shows the margin m_0 (left axis) and the dashed line the illiquidity $|\Lambda_0|$ (right axis) for different levels of speculator funding W_0 . Low dealer wealth leads to high illiquidity $|\Lambda_0|$ at time 0 and high liquidity risk going forward, implying high current margins m_0 . Hence, illiquidity and margins comove even though the financier is informed in this example.

in Figure 8.

7 Conclusion and New Testable Predictions

As discussed in the introduction, our model's market liquidity predictions are consistent with a host of stylized empirical facts. Our analysis further suggests a novel line of empirical work that tests the model at a deeper level: namely its prediction that speculators' funding is a driving force underlying these market liquidity effects.

First, it would be of interest to empirically study the determinants of margin requirements, e.g. using data from futures markets or from prime brokers. Our model suggests that both fundamental volatility and liquidity-driven volatility affects margins. Empirically, fundamental volatility can be captured using price changes over a longer time period, and the total fundamental and liquidity-based volatility is captured by short-term price changes as in the literature on variance ratios (see e.g. Campbell, Lo, and MacKinlay (1997)). Our model predicts that, in markets where it is harder for financiers to be informed, margins depend on the total fundamental and liquidity-based volatility. In particular, in times of liquidity crisis, margins increase in such markets, and, more generally, margins should comove with illiquidity in the time series and in the cross section.¹²

¹²One must be cautious with the interpretation of the empirical results related to changes in Regu-

Second, our model suggests that an exogenous shock to speculators' capital should lead to a reduction in market liquidity. Hence, a clean test of the model would be to identify exogenous capital shocks such as an unrelated decision to close down a trading desk, a merger leading to reduced total trading capital, or a loss in one market unrelated to the fundamentals of another market, and then study the market liquidity and margin around such events.

Third, the model implies that the effect of speculator capital on market liquidity is highly non-linear: a marginal change in capital has a small effect when speculators are far from their constraints, but a large effect when speculators are close from their constraints.

Fourth, the model suggests that a cause of the commonality in liquidity is that the speculator sector's shadow cost of capital is a driving state variable. Hence, a measure of speculator capital tightness should help explain the empirical comovement of market liquidity. Further our result "commonality of fragility" suggest that especially sharp liquidity reductions occur simultaneously across several assets.

Fifth, the model predicts that the sensitivity of margins and market liquidity to speculator capital is larger for securities that are risky and illiquid on average. Hence, the model suggests that a shock to speculator capital would lead to a reduction in market liquidity through a spiral effect that is stronger for illiquid securities.

Sixth, speculators are predicted to have negatively skewed returns since, when they hit their constraints, they make significant losses because of the endogenous liquidity spirals, and, in contrast, their gains are not amplified when prices return to fundamentals. This leads to conditional skewness and unconditional kurtosis of security prices.

Finally, our analysis suggests that central banks can help mitigate market liquidity problems by controlling funding liquidity. If a central bank is better than typical financiers of speculators at distinguishing liquidity shocks from fundamental shocks, then the central bank can convey this information and urge financiers to relax their funding requirements — as the Federal Reserve Bank of New York did during the 1987 stock market crash. Central banks can also improve market liquidity by boosting speculators' funding conditions during a liquidity crisis, or by simply stating the intention to provide extra funding during times of crisis, which will loosen margin requirements immediately as financiers' worst-case scenarios improve.

A Appendix: Proofs

Proof of Propositions 1 and 2. These results follow from the calculations in the

lation T since this regulation may not affect speculators but affects the demanders of liquidity, namely the customers.

text. ■

Proof of Proposition 3. We prove the proposition for $Z_1 > 0$, implying $p_1 \leq v_1$ and $x_1 \geq 0$. The complementary case is analogous. To see how the equilibrium depends on the exogenous shocks, we first combine the equilibrium condition $x_1 = -\sum_{k=0}^1 y_1^k$ with the speculators' funding constraint to get

$$m_1^+ \left(Z_1 - \frac{2}{\gamma(\sigma_2)^2} (v_1 - p_1) \right) \leq b_0 + p_1 x_0 + \eta_1 \quad (31)$$

that is,

$$m_1^+ \left(Z_1 - \frac{2}{\gamma(\sigma_2)^2} (v_1 - p_1) \right) - p_1 x_0 - b_0 \leq \eta_1 \quad (32)$$

For η_1 large enough, this inequality is satisfied for $p_1 = v_1$, that is, it is a stable equilibrium that the market is perfectly liquid. For η_1 low enough, the inequality is violated for $p_1 = \frac{2v_1}{\gamma(\sigma_2)^2} - Z_1$, that is, it is an equilibrium that the speculator is in default. We are interested in intermediate values of η_1 . If the left-hand-side of (32) is increasing in p_1 then p_1 is a continuous increasing function of η_1 , implying no fragility with respect to η_1 .

Fragile arises if the left-hand-side of (32) can be decreasing p_1 . Intuitively, this expression measures the speculators' funding need at the equilibrium position, and fragility arises if the funding need is greater when prices are lower, that is, further from fundamentals. (This can be shown to be equivalent to a non-monotonic excess demand function.)

When the financier is informed, the left-hand-side of (32) is

$$(\bar{\sigma} + \bar{\theta}|\Delta v_1| + p_1 - v_1) \left(Z_1 + \frac{2}{\gamma(\sigma_2)^2} (p_1 - v_1) \right) - p_1 x_0 - b_0 \quad (33)$$

Since the first product is a product of two positive increasing functions of p_1 , this is decreasing in p_1 only if x_0 is large enough.

When the financier is uninformed and $a = 0$, the left-hand-side of (32) is

$$(\bar{\sigma} + \bar{\theta}|\Delta p_1|) \left(Z_1 + \frac{2}{\gamma(\sigma_2)^2} (p_1 - v_1) \right) - p_1 x_0 - b_0 \quad (34)$$

When $p_1 < p_0$, $|\Delta p_1| = p_0 - p_1$ is decreasing in p_1 and, if $\bar{\theta}$ is large enough, this can make the entire expression decreasing. Also, the expression is decreasing if x_0 is large enough.

It can be shown that the price cannot be chosen continuous in η_1 when the left-hand-side of (32) can be decreasing. ■

Proof of Proposition 4. When the funding constraint binds, we use the implicit function theorem to compute the derivatives. Using that y_1 is given by (12), the equilibrium condition $x_1 = -y_1$, the fact that the speculators' funding constraint binds in an illiquid equilibrium, and that $v_1 - p_1 > 0$ when $Z_1 > 0$, we have

$$m_1^+ \left(Z_1 - \frac{2}{\gamma(\sigma_2)^2} (v_1 - p_1) \right) = b_0 + p_1 x_0 + \eta_1 \quad (35)$$

We differentiate this expression to get

$$\frac{\partial m_1^+}{\partial p_1} \frac{\partial p_1}{\partial \eta_1} \left(Z_1 - \frac{2}{\gamma(\sigma_2)^2} (v_1 - p_1) \right) + m_1^+ \frac{2}{\gamma(\sigma_2)^2} \frac{\partial p_1}{\partial \eta_1} = \frac{\partial p_1}{\partial \eta_1} x_0 + 1 \quad (36)$$

which leads to Equation (23) after rearranging. The case of $Z_1 < 0$ and Equation (24) is analogous.

Finally, spiral effects happen if one of the last two terms in the denominator of the right-hand-side of Equations (23)-(24) is negative. When the speculator is informed, $\frac{\partial m_1^+}{\partial p_1} = 1$ and $\frac{\partial m_1^-}{\partial p_1} = -1$ using Proposition 1. Hence, in this case margins are stabilizing.

If the speculators are uninformed and a approaches 0, then using Proposition 2 we have that $\frac{\partial m_1^+}{\partial p_1} = \frac{\partial m_1^+}{\partial \Lambda_1}$ approaches $-\bar{\theta} < 0$ for $v_1 - v_0 + \Lambda_1 - \Lambda_0 < 0$ and $\frac{\partial m_1^-}{\partial p_1} = \frac{\partial m_1^-}{\partial \Lambda_1}$ approaches $\bar{\theta} > 0$ for $v_1 - v_0 + \Lambda_1 - \Lambda_0 > 0$. This means that there is a margin spiral with positive probability. The case of a loss spiral is immediately seen to depend on the sign on x_0 . ■

Proof of Proposition 5. We first consider the equation that characterizes a constrained equilibrium. When there is selling pressure from customers, $Z_1^j > 0$, it holds that

$$|\Lambda_1^j| = -\Lambda_1^j = v_1^j - p_1^j = \min \left\{ \phi_1 m_1^{j+}, \frac{\gamma(\sigma_2^j)^2}{2} Z_1^j \right\} \quad (37)$$

and if customers are buying, $Z_1^j < 0$, we have

$$|\Lambda_1^j| = \Lambda_1^j = p_1^j - v_1^j = \min \left\{ \phi_1 m_1^{j-}, \frac{\gamma(\sigma_2^j)^2}{2} (-Z_1^j) \right\} \quad (38)$$

Using the equilibrium condition $x_1^j = -\sum_k y_1^{j,k}$, Equation (12) for $y_1^{j,k}$, the speculators' funding condition becomes

$$\sum_{Z_1^j > \frac{2\phi_1 m_1^{j+}}{\gamma(\sigma_2^j)^2}} m_1^{j+} \left(Z_1^j - \frac{2\phi_1 m_1^{j+}}{\gamma(\sigma_2^j)^2} \right) + \sum_{-Z_1^j > \frac{2\phi_1 m_1^{j-}}{\gamma(\sigma_2^j)^2}} m_1^{j-} \left(-Z_1^j - \frac{2\phi_1 m_1^{j-}}{\gamma(\sigma_2^j)^2} \right) = \sum_j x_0^j p_1^j + b_0 + \eta_1 \quad (39)$$

where the margins are evaluated at the prices solving (37)–(38). When ϕ_1 approaches infinity, the left-hand-side of (39) becomes zero, and when ϕ_1 approaches zero, the left-hand-side approaches the capital needed to make the market perfectly liquid. As in the case of one security, there can be multiple equilibria and fragility (Proposition 3). On a stable equilibrium branch, ϕ_1 increases as η_1 decreases.

Of course, the equilibrium shadow cost of capital ϕ_1 is random since $\eta_1, \Delta v_1^1, \dots, \Delta v_1^J$ are random. To see the commonality in liquidity, we note that $|\Lambda^j|$ is increasing in ϕ_1 for each $j = k, l$. To see this, consider first the case $Z_1^j > 0$. When the financiers are uninformed, $a = 0$, and $\theta^j = 0$, then $m_1^{j+} = \bar{\sigma}^k$, and, therefore, Equation (37) shows directly that $|\Lambda_1^j|$ increases in ϕ_1 (since the minimum of increasing functions is increasing). When financiers are informed and $\theta^j = 0$ then $m_1^{j+} = \bar{\sigma}^k + \Lambda_1^j$, and, therefore, Equation (37) can be solved to be $|\Lambda_1^j| = \min\{\frac{\phi_1}{1+\phi_1}\bar{\sigma}^j, \frac{\gamma(\sigma_2^k)^2}{2}Z_1^k\}$, which increases in ϕ_1 . Similarly, Equation (38) shows that $|\Lambda^j|$ is increasing in ϕ_1 when $Z_1^j < 0$.

Now, since $|\Lambda^j|$ is increasing in ϕ_1 and does not depend on other states variables under these conditions, $\text{Cov}[\Lambda^k(\phi), \Lambda^l(\phi)] \geq 0$ because any two functions which are both increasing in the same random variable are positively correlated.

To see part (ii) of the proposition, note that, for all j , $|\Lambda^j|$ is a continuous function of ϕ_1 , which is locally insensitive to ϕ_1 if and only if the speculator is not marginal on security j (i.e. if the second term in Equation (37) or (38) attains the minimum). Hence, $|\Lambda^j|$ jumps if and only if ϕ_1 jumps.

To see part (iii), we write illiquidity using Equations (37)–(38) as

$$|\Lambda_1^j| = \min\left\{\phi_1 m_1^{j, \text{sign}(Z_1^j)}, \frac{\gamma(\sigma_2^j)^2}{2}|Z_1^j|\right\} \quad (40)$$

Hence, using the expression for the margin, if the financier is uninformed and $\theta^j = a = 0$ then

$$|\Lambda_1^j| = \min\left\{\phi_1 \bar{\sigma}_1^j, \frac{\gamma(\sigma_2^j)^2}{2}|Z_1^j|\right\} \quad (41)$$

and, if the financier is informed and $\theta^j = 0$, then

$$|\Lambda_1^j| = \min\left\{\frac{\phi_1}{1+\phi_1}\bar{\sigma}_1^j, \frac{\gamma(\sigma_2^j)^2}{2}|Z_1^j|\right\} \quad (42)$$

In case of an uninformed financier as in (41), we have that if $x_1^k \neq 0$

$$|\Lambda_1^k| = \phi_1 \bar{\sigma}_1^k \geq \phi_1 \bar{\sigma}_1^l \geq |\Lambda_1^l| \quad (43)$$

and if $|Z_1^k| \geq |Z_1^l|$

$$|\Lambda_1^k| = \min\left\{\phi_1 \bar{\sigma}_1^k, \frac{\gamma(\sigma_2^k)^2}{2}|Z_1^k|\right\} \geq \min\left\{\phi_1 \bar{\sigma}_1^l, \frac{\gamma(\sigma_2^l)^2}{2}|Z_1^l|\right\} = |\Lambda_1^l| \quad (44)$$

With an informed financier, if it is seen that $|\Lambda_1^k| \geq |\Lambda_1^l|$ using the same arguments.

For part (iv) of the proposition, we use that

$$\frac{\partial |\Lambda_1^j|}{\partial(-\eta_1)} = \frac{\partial |\Lambda_1^j|}{\partial \phi_1} \frac{\partial \phi_1}{\partial(-\eta_1)} \quad (45)$$

Further, $\frac{\partial \phi_1}{\partial(-\eta_1)} \geq 0$ and, from Equations (41)–(42) we see that $\frac{\partial |\Lambda_1^k|}{\partial \phi_1} \geq \frac{\partial |\Lambda_1^l|}{\partial \phi_1}$. The result that $\text{Cov}(\Lambda^k, \phi) \geq \text{Cov}(\Lambda^l, \phi)$ now follows from Lemma 1 below. ■

Lemma 1 *Let X be a random variable and g_i , $i = 1, 2$, be weakly increasing functions X , where g_1 has a larger derivative than g_2 , that is, $g_1'(x) \geq g_2'(x)$ for all x . Then,*

$$\text{Cov}[X, g_1(X)] \geq \text{Cov}[X, g_2(X)] \quad (46)$$

Proof. For $i = 1, 2$ we have

$$\text{Cov}[X, g_i(X)] = E[(X - E[X])g_i(X)] \quad (47)$$

$$= E\left[(X - E[X])\left(\int_{E[X]}^X g_i'(y)dy\right)\right] \quad (48)$$

The latter expression is a product of two terms that always have the same sign. Hence, this is higher if g_i' is larger. ■

Liquidity Risk (Time 0).

To see that the speculator's time 0 problem is concave, let us first focus on the case in which the speculator goes long a single asset $x_0^+ > 0$. He maximizes $E_0[\phi_1 W_1]$ subject to his margin constraint, where his ϕ_1 is either a constant that depends on future price realizations $\frac{E[v_1] - p_1}{m_1}$ if his wealth $W_1 > 0$ or 1 if he goes bankrupt, i.e. $W_1 = W_0 + (p_1 - p_0)x_0^{+,i} \leq 0$. Taking the first order condition of the corresponding Lagrangian with respect to x_0^+ yields

$$\int_{W_0^i + (p_1 - p_0)x_0^+ \geq 0} \phi_1 (p_1 - p_0) dF(p_1) - (\phi_0^i - E_0[\phi_1^i])m_0^+ = 0, \quad (49)$$

where we set $(\phi_0 - E_0[\phi_1])$ to be the Lagrange multiplier for constraint (4), such that ϕ_0 reflects the marginal value of an extra dollar for speculator i at time $t = 0$. Note that the lower bound of the integral (over different p_1 -realizations) is $p_0 - \frac{W_0}{x_0^+}$. Hence, the integral term decreases in x_0^+ , implying that the second derivative of the Lagrangian is negative. After solving the analogous problem for the optimal short position x_0^- , one takes the maximum over these two local maxima.

Section 6 focuses on the case of speculators who are unconstrained at time $t = 0$. When the speculator is constrained is time 0, then invests only in securities with the

highest expected profit per capital use, where profit is calculated using the pricing kernel $\phi_1^i/E_0[\phi_1^i]$. In this case, his time-0 shadow cost of capital is

$$\phi_0^i = E_0[\phi_1^i] \left(1 + \max_j \left\{ \frac{E_0[\frac{\phi_1^i}{E_0[\phi_1^i]} p_1^j] - p_0^j}{m_0^+}, -\frac{E_0[\frac{\phi_1^i}{E_0[\phi_1^i]} p_1^j] - p_0^j}{m_0^-} \right\} \right) \quad (50)$$

■

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