Fun Facts With Returns

- i.i.d. monthly returns with mean $\bar{R}$ and variance σ^2.
- Then
$$\bar{R}_y = 12\bar{R}.$$
and
$$\sigma_y^2 = E\left[\sum_{i=1}^{12}(R_i - \bar{R})\right]^2 = E\left[\sum_{i=1}^{12}(R_i - \bar{R})^2\right] = 12\sigma^2$$
by independence.

- So
$$\bar{R} = \frac{1}{12}\bar{R}_y$$
and
$$\bar{\sigma} = \frac{1}{\sqrt{12}}\bar{\sigma}_y.$$
- Of course, this analysis can be extended to any period length p as a fraction of a year:
$$\bar{R} = p\bar{R}_y$$
and
$$\bar{\sigma} = \sqrt{p}\bar{\sigma}_y.$$
The square root term is the problem.

The Effect of Period Length

- The expected return is proportional to the period length.
- The standard deviation is proportional to the square root of period length.
- The estimate of the ratio of the two: mean over standard deviation increases dramatically as the period is reduced, because the estimate of s.d. doesn't fall that fast.
- Rates of returns have very high standard deviations for small estimation periods.

Example

- Let's take $\bar{R}_y = 12\%$, $\sigma_y = 15\%$.
- So $\bar{R}_{1/12} = 1\%$, $\sigma_{1/12} = 4.33\%$ for the corresponding monthly values.
- The s.d. of the monthly return is 4.3 times the expected rate of return: for the yearly figures, the ratio is 1.25.
- For daily data: $p = 1/250$ and $\bar{R}_{1/250} = .048\%$, $\sigma_{1/250} = .95\%$ and the ratio is 19.8.
- This confirms ordinary experience: stocks can move a lot in a day even though their expected returns are small.

Mean Blur

This makes estimation of means nearly impossible.

- Start with monthly returns.
- Assume stationarity: $\bar{R}$, σ^2 for the monthly returns.
- Assume sample of length n.
- Then our estimate:
$$\hat{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$$
and the expected value is equal to $\bar{R}$.
- Now, what about the standard deviation of our estimate for $\bar{R}$
$$\sigma^2_{\hat{R}} = E[\hat{R} - \bar{R}]^2 = E\left[\frac{1}{n}\sum_{i=1}^{n}(R_i - \bar{R})\right]^2 = \frac{1}{n}\sigma^2.$$

- Hence:
$$\sigma_{\hat{R}} = \frac{\sigma}{\sqrt{n}},$$
which is the formula for the error in the estimate of the mean.

Plugging In

- With 12 months of data, $\bar{R}_{1/12} = 1\%$, $\sigma_{1/12} = 4.33\%$ so that
$$\sigma_{\hat{R}} = \frac{4.33\%}{\sqrt{12}} = 1.25\%,$$
which is larger than the mean itself.
- So, how much data is enough? Using 1 year of data we say that the mean monthly return is 1 percent plus or minus 1.25 percent.
- What about 4 years? Then we cut the standard deviation down by a factor of 2 - which is not good.
- To get a good estimate, we need a standard deviation that's more like a tenth of the mean. That requires $n = 43.3^2 = 1875$ or 156 years of data. item But then the mean and the variance

are not likely to be the same in 156 years.

- You just can't measure means using historical data. Increasing frequencies doesn't help. For homework, do the calculation for daily returns (250 per year) and see what you get. Then, try it again for semi-annual periods. Any better?
- The point is that high frequency data does not help when you are estimating the mean return.

Consequences for Empirical Work

- One place where this is a big problem is in the portfolio performance evaluation industry.
- Suppose you're trying to predict from past returns whether one portfolio is likely to outperform the other in the future.
- You can't tell with any confidence that the average return on one portfolio is any different from the average return on the other because the statistical technology for estimating the mean is so bad.

A statistical example (Luenberger)

- simulated 8 years of monthly data with mean 1% and s.d. 4.33%.
- This corresponds to annual returns of 12% with s.d. of 15%.
- The results look like this:

yr:	1	2	3	4	5	6	7	8	overall
$\hat{R}$	3.02	.52	1.67	0.01	1.76	2.06	1.37	.17	1.32
σ	5.01	5.88	3.21	3.81	2.98	3.24	4.66	3.55	4.12

Estimating other moments

Is a lot easier. For example, with σ:

- Sample size n of returns R_i:

$$\hat{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \hat{R}_i)^2$$

- and the var of the estimate of the variance is

$$var(s^2) = \frac{2\sigma^4}{n-1}$$

so that

$$s.d.(s^2) = \frac{\sqrt{2}\sigma^2}{\sqrt{n-1}}$$

- This shows that the s.d. of the variance is $\frac{\sqrt{2}}{\sqrt{n-1}}$ times the variance and the relative error in estimating σ^2 is not too huge if n is big enough.

Other Practical Consequences

Estimating α and β in a market model

$$Z_i = \hat{\alpha}_i + \hat{\beta}_i Z_m$$

where the Z are excess returns. β is estimated well enough, but not α.

Put another way, estimates of superior performance, as measured by Jensen's α, are bad.

What's the Mean Telling Us?

This is not a trick question, but has to do with the distribution of long–run returns.

- Suppose you took a time series of stock returns with a an annual mean of 8% and variance of 50%. Hold the stock for 100 years. How much money would you expect to have?
- That's right, it's a lot.
- Now, let's ask a slightly different question: what's the chance you'll lose your shirt? Pretty low, right?
- You're diversifying across time, right?
- Let's see if that's true. Consider the following spreadsheet: