CAPM Review

Chapter 5: Tests of the CAPM

- CAPM Review (Ingersoll ch.4)
  - Single-period model.
  - Investors are mean-variance maximizers, minimizing variance subject to a mean constraint.
    - They may have mean-variance preferences (i.e. quadratic).
    - The distribution of returns may be such that all risk-averse investors act as if they have mean-variance preferences (e.g. normally distributed returns).

Justifications

- To see this, look at a Taylor series expansion of utility of end-of-period wealth (around expected end-of-period wealth):

$$u(\tilde{W}) = u(E(\tilde{W})) + u'(E(\tilde{W}))(\tilde{W} - E(\tilde{W})) + \frac{1}{2}u''(E(\tilde{W}))(\tilde{W} - E(\tilde{W}))^2 + R_3$$

- where

$$R_3 = \sum_{n=3}^{\infty} \frac{1}{n!} u^n(E(\tilde{W}))(\tilde{W} - E(\tilde{W}))^n$$

Justifications – Cont…

- If the series converges, the differentiation and expectation operations are interchangeable, and expected utility is then (taking expectations of both sides):

$$E(u(\tilde{W})) = u(E(\tilde{W})) + \frac{1}{2}u''(E(\tilde{W}))\sigma^2(W) + E(R_3)$$

- where

$$E(R_3) = \sum_{n=3}^{\infty} \frac{1}{n!} u^n(E(\tilde{W}))m^n(\tilde{W})$$

- and $m^n(\widetilde{W})$ is the n$^{\text{th}}$ central moment of $\widetilde{W}$.

Justifications – Cont…

- For arbitrary return distributions, the mean-variance model can be motivated by choice of quadratic utility.
- In this case, $R_3 = 0$ so that

$$\begin{aligned} E(u(\tilde{W})) &= E(\tilde{W}) - \frac{b}{2}E(\tilde{W}^2) \\ &= E(\tilde{W}) - \frac{b}{2}\left(E(\tilde{W})^2 + \sigma^2(\tilde{W})\right) \end{aligned}$$

- When expected returns and variances are finite, quadratic utility is sufficient for mean-variance preferences.

What's Wrong with Quadratic Preferences?

- Increasing absolute risk aversion – wealthier individuals invest a smaller amount in the risky asset.
  - Recall, Arrow-Pratt measure of absolute risk aversion is: $-\frac{u''}{u'}$
  - The higher it is the more curved is the utility function.
  - Risky investments are not normal goods.
- Satiation (marginal utility of wealth decreases after a point).

Returns are Multivariate Normally Distributed

- Then, third and higher moments can be expressed as functions of the first two.
- Normal distributions are stable under addition, so that portfolios of assets with normally distributed returns are normally distributed.
- Lognormal distributions can also be described by two moments but are not stable under addition.
- Normal distributions are not bounded from below.
- What do you do about options?

The Economy

- $N \geq 2$ risky assets, with no redundant securities.
  - A finite mean vector μ (Nx1).
  - Variance-covariance matrix $\boldsymbol{\Omega}$ (NxN), that is nonsingular and symmetric ($\sigma_{ij} = \sigma_{ji}$).
- Portfolio vector ω_a (Nx1)
  - Mean: $\mu_a = \omega_a' \mu$,
  - Variance $\sigma_a^2 = \omega_a' \boldsymbol{\Omega} \omega_a$.
  - The covariance between the returns on portfolios a and b is $\sigma_{ab} = \omega_a' \boldsymbol{\Omega} \omega_b$.

Two Assets, 1 and 2

- Means, portfolios, V/C matrix:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$$

The Investor's Problem

- Minimize portfolio variance subject to a mean constraint (minimum-variance portfolios), or
- Maximize mean for a fixed portfolio variance (mean-variance efficient portfolios).
- Second is what we want to work with, first is analytically easier.
- We consider two cases:
  - No risk-free asset (Black CAPM).
  - Risk-free asset (Standard CAPM).

With No Risk-free Asset

$$Min_{\omega}\ \omega'\Omega\omega \qquad (1)$$

- Subject to

$$\omega'\mu = \mu_p, \qquad (2)$$

$$\omega'1 = 1. \qquad (3)$$

Solution Procedure

- The Lagrangian function, *L*, is given by:

$$L = \omega'\Omega\omega + \delta_1(\mu_p - \omega'\mu) + \delta_2(1 - \omega'1) \qquad (4)$$

  where δ_1 and δ_2 are Lagrange multipliers.
- Differentiating with respect to the *N* portfolio weights yields *N* first-order conditions:

$$2\Omega\omega - \delta_1\mu - \delta_2 1 = 0 \qquad (5)$$

$$(N \times N) * (N \times 1) - (N \times 1) = (N \times 1)$$

Solution

- Combining (5), (2), and (3) yields the solution:

$$\omega_p = g + h\mu_p \qquad (6)$$

- Where **g** and **h** are Nx1 vectors given by:

$$\mathbf{g}_{(Nx1)} = \frac{1}{D}\left[B\mathbf{\Omega}^{-1}\mathbf{1} - A\mathbf{\Omega}^{-1}\mu\right] \qquad (7)$$

$$\mathbf{h}_{(Nx1)} = \frac{1}{D}\left[C\mathbf{\Omega}^{-1}\mu - A\mathbf{\Omega}^{-1}\mathbf{1}\right] \qquad (8)$$

$$A = \mathbf{1}'\mathbf{\Omega}^{-1}\mu; \quad B = \mu'\mathbf{\Omega}^{-1}\mu$$

$$C = \mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1}; \quad D = BC - A^2$$

The Equation of the MV Set

- We know that

$$\sigma^2 = \omega'\Omega\omega = \omega'\Omega(g + h\mu_p)$$

- Substituting in for **g** and **h** yields:

$$\sigma^2 = \frac{C\mu^2 - 2A\mu + B}{D}$$

- This is the equation of a parabola. In mean-standard deviation space, the curve is a hyperbola.

MV Set Without a Risk-free Asset

- Draw the picture.
- Identify portfolios "d" and "g".
- The asymptotes: $\mu = A/C \pm \sigma(D/C)^{1/2}$.

Back to Two Assets

- The Lagrangian function, *L*, is given by:

$$
\begin{aligned}
L &= \omega' \mathbf{\Omega} \omega + \delta_1(\mu_p - \omega' \mu) + \delta_2(1 - \omega' \mathbf{1}) \\
&= \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1 \omega_2 \sigma_{12} + \\
&\quad \delta_1(\mu_p - \omega_1 \mu_1 - \omega_2 \mu_2) + \delta_2(1 - \omega_1 - \omega_2)
\end{aligned}
$$

Two Assets – Cont…

- There are two first-order conditions, or only one if one plugs in from the portfolio constraint: $\omega_2 = 1 - \omega_1$.
- Differentiating yields:

$$\omega_1: \quad 2\omega_1\sigma_1^2 + 2\omega_2\sigma_{12} - \delta_1\mu_1 - \delta_2 = 0$$

$$\omega_2: \quad 2\omega_2\sigma_2^2 + 2\omega_1\sigma_{12} - \delta_1\mu_2 - \delta_2 = 0$$

- As well as

$$\delta_1: \quad \mu_p - \omega_1\mu_1 - \omega_2\mu_2 = 0$$

$$\delta_2: \quad 1 - \omega_1 - \omega_2 = 0$$

- Solving yields the result. Because there are two assets, the interior of the hyperbola is empty.

Efficient Set Results

- **Result 1:** The minimum variance frontier can be generated from any two minimum variance portfolios. (Two fund separation)
- **Result 1':** Any portfolio of minimum variance portfolios is also minimum variance.
- **Result 2:** Let the returns on two MV portfolios be given by R_p and R_r. Then:

$$\sigma_{rp} = \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)\left(\mu_r - \frac{A}{C}\right) + \frac{1}{C} \qquad (9)$$

Efficient Set Results – Cont…

- **Result 3:** Let **g** be the global minimum variance portfolio. Then:

$$\omega_g = \frac{1}{C}\Omega^{-1}1 \qquad (10)$$

$$\mu_g = \frac{A}{C} \qquad (11)$$

$$\sigma_g^2 = \frac{1}{C} \qquad (12)$$

Efficient Set Results – Cont…

- **Result 4:** For each minimum-variance portfolio *p*, except *g*, there exists a unique portfolio that has zero covariance with it. This is the “zero-beta” (0p) portfolio with respect to *p*.
- **Result 4':** Let *a* be any (not necessarily MV) portfolio. Then:

$$\sigma_{ga} = \frac{1}{C}$$

MV Portfolios Without a Risk-free Asset

- Result for empirical work:
  - **Result 5:** Consider the following multiple regression of (any) return R_a on the return on any MV portfolio R_p (except R_g)and R_{0p}. That is:

$$R_a = \beta_0 + \beta_1 R_{0p} + \beta_2 R_p + \varepsilon_p \qquad (14)$$

where

$$E[\varepsilon_p \mid R_p, R_{0p}] = 0. \qquad (15)$$

Regression Coefficients

$$\beta_2 = \frac{\sigma_{ap}}{\sigma_p^2} \equiv \beta_{ap} \qquad (16)$$

$$\beta_1 = \frac{\sigma_{a0p}}{\sigma_{0p}^2} \equiv 1 - \beta_{ap} \qquad (17)$$

$$\beta_0 = 0. \qquad (18)$$

Efficient Set Results – Cont…

- **Result 5':**

$$\mu_a = (1-\beta_{ap})\mu_{0p} + \beta_{ap}\mu_p \qquad (19)$$

- The zero beta rate can be treated as a parameter to be estimated providing a cross-sectional restriction on the asset-specific intercept.

  Next we add a risk-free asset to the economy with return R_f.

With a Risk-free Asset

- The agent's problem:

$$Min_{\omega}\ \omega'\Omega\omega \qquad (20)$$

subject to

$$\omega'\mu + (1-\omega'1)R_f = \mu_p \qquad (21)$$

Solution Procedure

- The Lagrangian function, *L*, is now given by:

$$L = \omega'\Omega\omega + \delta(\mu_p - \omega'\mu - (1 - \omega'1)R_f), \qquad (22)$$

where δ is the Lagrange multiplier.

- Differentiating yields *N* first-order conditions:

$$2\Omega\omega - \delta\mu - R_f 1 = 0. \qquad (23)$$

- Combining (23), (21) yields the solution:

$$\omega_p = \frac{(\mu_p - R_f)}{(\mu - R_f 1)'\Omega^{-1}(\mu - R_f 1)}\Omega^{-1}(\mu - R_f 1) \qquad (24)$$

Solution Procedure – Cont…

- We can write ω_p as a scalar that depends on the mean of p times a portfolio weight vector that does not depend on p:

$$\omega_p = c_p \overline{\omega} \qquad (25)$$

- where

$$c_p = \frac{(\mu_p - R_f)}{(\mu - R_f 1)'\Omega^{-1}(\mu - R_f 1)} \qquad (26)$$

- and

$$\overline{\omega} = \Omega^{-1}(\mu - R_f 1) \qquad (27)$$

Result

- Thus, with a risk-free asset, all MV portfolios are combinations of a given risky asset portfolio, with weights proportional to $\overline{\omega}$, and the risk-free asset. This portfolio of risky assets is called the tangency portfolio and has weight vector:

$$\omega_q = \frac{1}{1'\Omega^{-1}(\mu - R_f 1)}\Omega^{-1}(\mu - R_f 1) \qquad (28)$$

Result

- Equation (28) divides the elements of $\overline{\omega}$ to get a vector whose elements sum to one – a portfolio weight vector. With a risk-free asset, all efficient portfolios lie on a line from R_f that runs through μ_q.

MV Portfolios with R_f

- Draw the picture.
- Identify tangency portfolio and risky asset only frontier.

Efficient Set Results

- **Result 5' with risk free asset:**

$$\mu_a - R_f = \beta_{aq}(\mu_q - R_f)$$

- The relation between expected excess returns on any asset is linearly related to the asset's beta with the tangency portfolio, q.
- This relation holds for any efficient portfolio (not only q).

Relation to CAPM

- Note that the linear pricing results obtain for any efficient portfolio
- The economic content of the CAPM is purely in arguing that in equilibrium the market portfolio is the tangency portfolio, q.
- Roll's (1977) critique (read this).

Relation to CAPM

- If investors have concave derived utility functions of the form:

$$V(\mu,\sigma^2),\ V_2 < 0,\ V_1 > 0$$

- All investors have homogeneous beliefs about the distribution of asset returns.
- The riskless asset can be bought and sold in unlimited amounts.

CAPM Equilibrium

- Investor's problem:

$$Max\ V(R_f + \omega'(\mu - R_f 1), \omega' \Sigma \omega)$$

- First Order Conditions:

$$0 = V_1(\bullet)(\mu - R_f 1) + V_2(\bullet)\Sigma\omega$$

- Solution:

$$\omega^* = \frac{-V_1}{2V_2}\Sigma^{-1}(\mu - R_f 1)$$

CAPM Equilibrium

- Note that the optimal portfolio weights:

$$\omega^* = \frac{-V_1}{2V_2}\Sigma^{-1}(\mu - R_f 1)$$

- are proportional to the tangency portfolio and the rest is invested in the risky asset.
- Since all investors hold combinations of the tangency portfolio and the risk-free asset the aggregate demand for each risky asset must be in proportion to its representation in the tangency portfolio

CAPM Equilibrium

- In equilibrium, demand equals supply so that the supply of each risky asset must be in proportion to its weight in the tangency portfolio: $\omega_M = \omega_q$

- where the weight in the market portfolio is the value of the risky asset divided by the total value of all risky assets.

Sharpe Ratio

- The expected excess return per unit risk. It is defined as the mean excess return divided by its standard deviation:

$$sr_a = \frac{\mu_a - R_f}{\sigma_a} \qquad (29)$$

- Sharpe ratios in the diagram.
- The tangency portfolio is the portfolio with the maximum Sharpe ratio of all portfolios of risky assets.