Testing The CAPM

Statistical Framework for Estimation and Testing

The Two Models

- First, we will consider the Sharpe-Lintner CAPM:
  - Let investors borrow and lend at the risk-free rate, R_f.
- Later, we will consider the Black CAPM:
  - In this case, recall, the return on the zero-beta asset is another parameter to be estimated.

Testing Strategy

- We're going to start with the market model, which says, for each stock:

$$E[R_{it}] = \alpha_i + \beta_i E[R_{mt}]$$

- Now, think about the market model in *excess return* form:

$$E[Z_{it}] = \alpha_i + \beta_i E[Z_{mt}]$$

where $Z_{it} = R_{it} - R_f$ and $Z_{mt} = R_{mt} - R_f$.

- Obviously, the α_i in the two models are different.

Testing Strategy cont…

- This is the "unrestricted model" that we will estimate.
- Then, we impose the restrictions on this model that are implied by the various forms of the CAPM.
- These are the restrictions we will be testing.

Sharpe-Lintner CAPM Tests

- Z_t is an Nx1 vector of excess returns for the N assets. Then, excess returns can be described using the following "market model" regression:

$$\mathbf{Z}_t = \alpha + \beta Z_{mt} + \varepsilon_t$$
$$E[\varepsilon_t] = 0$$
$$E[\varepsilon_t \varepsilon_t'] = \Sigma$$
$$E[Z_{mt}] = \mu_m; \quad E[(Z_{mt} - \mu_m)^2] = \sigma_m^2$$
$$Cov[Z_{mt}, \varepsilon_t] = 0$$

- We redefine μ to refer to expected excess return.

Discussion

- The above model is just a "stacked" version of the market model written in excess return form.
- Now, think about the restrictions that the CAPM imposes. The CAPM says that:

$$E[R_{it}] - R_f = \beta_i (E[R_{mt}] - R_f)$$
$$E[Z_{it}] = \beta_i E[Z_{mt}]$$

- Essentially, this means that: $\alpha_i = 0\ \forall i$.
- What do these restrictions imply for the stacked system?

Testable Implications of the S-L CAPM

- Linearity: If the reference portfolio is mean-variance efficient, then linear pricing obtains. The CAPM implies that the market portfolio is mean-variance efficient.
- This is true for both the Black and S-L versions of the CAPM.
- Further, if there exists a risk-free asset, so that the S-L version of the CAPM holds, then the market portfolio must be (proportional to) the tangency portfolio.
- So what we're going to do is estimate N alphas and N betas and then impose the CAPM restrictions that the N alphas must be zero.

Testable Implications – Operationally

- All elements of the vector α are *simultaneously* equal *and* equal to zero.
  - This is the principal hypothesis for tests of the Sharpe-Lintner CAPM.
  - If all elements of the vector α are simultaneously zero, then the portfolio *m* is the tangency portfolio.
  - Keep this in mind when looking at the GRS (1989) test statistic as a test of the efficiency of the "market" portfolio.

Discussion of Testable Implications

- The CAPM predicts that all investors hold portfolios that are efficient in expected return – standard deviation space.
- By the two fund theorem, the market portfolio is expected to be efficient.
- For the CAPM to be true, the market portfolio must lie in the efficient set. This is in fact the economic implication of the CAPM analysis, the "identification" of an efficient portfolio.

Implicit Assumptions

- Probability distributions for returns on stocks and bonds are stationary.
  - Hence, sample moments converge to population moments, so we can make statistical inferences.
- The estimates are subject to sampling error, so that the market portfolio is not likely to be *ex-post* efficient.
- The question is: Can we distinguish statistically whether the market was *ex-ante* efficient from observing *ex-post* data?

Data Issues

- Time interval for measuring returns (monthly, daily, etc.).
- Time interval for estimation (e.g. 5 years).
- Assets to include in test.
- The choice of a market portfolio.

Early Tests

- Focus of early tests:
  - Not direct tests of the efficiency of the index *m*.
  - Centered on properties of SML (mean/beta space).
  - Recall that for any MVE portfolio, there must be a linear, positive relation between β_i and $E[r_i]$. This is the SML:

$$E[r_i] = \alpha_i + \beta_i E[r_{MVE}]$$

  - This is essentially the market model.

Early Test Procedure

- Two pass technique:
  - First pass: time series estimation where security (or portfolio) returns were regressed against an index, *m*:

$$r_{it} = \alpha_i + \beta_i r_{mt}$$

  - Second pass: cross-sectional. Beta is related to average return:

$$E[r_i] = \gamma(1 - \beta_i) + \delta\beta_i$$

  - γ equals r_f in the S-L CAPM and r_{0m} in the Black CAPM. While δ is the expected market return.

BJS (1972)

- Test S-L CAPM
  - Data: 1926-1965 NYSE stocks.
  - *m* is the value-weighted NYSE index.
    - Why value-weighted?
  - Start with 1926-1930.
  - Calculate betas (pass 1).
  - Rank securities by beta and form into portfolios 1-10.
  - Compute monthly returns for each of the 12 months of 1931 for the 10 portfolios.
  - Do stage 2 for 1931 on the 10 portfolios.
  - Recompute betas using 1927-1931 period, and so on. (This is called a rolling regression.)

Combining Into Portfolios

- The problem with estimating betas is the "errors in variables" problem.
- Each stock's beta is estimated with error.
- In the second stage, the estimated betas are used as regressors, so that the estimation error biases the second-stage coefficient, γ, downward.
- If the estimation errors in the betas are uncorrelated across stocks, combining stocks into portfolios reduces estimation error, and increases the accuracy of the second stage regression.

Sorting by Beta

- Why do this?
  – The downside of combining into portfolios is the information loss.
    - Indeed, combining randomly into portfolios leads to portfolio betas close to 1.0.
    - Although portfolio betas are measured relatively accurately, the information loss is severe – there is little cross-sectional variation left in the independent variable.
    - Sorting by beta preserves cross-sectional variation in the second stage regression.

Rolling Regression

- Why do this?
  - Even after combining into portfolios, an estimated beta may be high for two reasons
    - The true portfolio beta is high.
    - The estimation error is positive.
  - To eliminate the bias in estimation error (that high beta portfolios have positive error and low beta portfolios have negative error), recompute the portfolio betas in a different period.

Results

- Risk premium is 12.97% per year.
- R_f is 6.225% per year – too high.
- Little or no evidence for non-linearity.

Fama-MacBeth (1973)

- Regressors:
  - If, for example, the joint distribution of asset returns is multivariate normal, then the joint distribution of the return on any security and the market is bivariate normal:

    $$R_i = \alpha_i + \beta_i R_M + \varepsilon_i, \quad i = 1,\ldots,N$$

    where

    $$\alpha_i = E[R_i] - \beta_i E[R_M]$$
  - Also include terms for non-linearity (β_i^2) and residual variance (σ_i^2). Why do it this way?

Regressors

- Since ε_i and R_M are independent,

  $$\sigma^2(R_i) = \beta_i^2 \sigma^2(R_M) + \sigma^2(\varepsilon_i)$$
- Now, consider a model where expected security returns are determined by a securities total variance, rather than β.
- Security variance is given above.
- Could put variance in directly or do as Fama and MacBeth have, putting in the components to allow the estimation to distinguish different effects.

Roll Critique

- If you use any *ex-post* MVE portfolio as the index, then the SML relationship will be exactly satisfied by the sample betas time series mean returns. (Just math.)
- Linearity is a consequence of MV mathematics.
- This wouldn't be a problem if you could identify the market portfolio. Then you could test whether the market was MVE, *ex-post*. But you don't know the true market portfolio.
- So then, all you can do is test whether your proxy is MVE.
- You can't choose your proxy on the basis of *ex-post* efficiency – that leads to a tautological test.

Time Series Tests

- First, consider the Sharpe-Lintner CAPM.
  – Start with the S-L CAPM derived from distributional assumptions.
    - Without distributional assumptions, testing the CAPM is harder.
    - Must rely on semi-parametric approach (like GMM).
    - For now we'll stick with multivariate normality.

The Maximum Likelihood Approach

- There are N assets and hence, N equations.
- For each equation, we can run OLS and obtain estimates of α_i and β_i, $I = 1,\ldots,N$.
- We could also estimate the equations jointly.
- Is there any advantage to doing this, that is, run the "seemingly unrelated" regression on the system?

Assignment

- Demonstrate (for yourself) that with identical regressors in every equation, that the point estimates and standard errors obtained from dong "seemingly unrelated" regressions are identical to those obtained when using equation-by-equation OLS. That is, there is no advantage in estimating the system of equations jointly. Assume of course, that the OLS assumptions for consistency and asymptotic normality are satisfied.

The Covariance Matrix

- As you will see, joint estimation is useless for estimating α's and β's.
- However, for our joint test, it's not useless. We need the covariance matrix for our joint test.

The Likelihood Function

- Given joint normality of excess returns, the likelihood function for the cross-section of excess returns in a single period is:

$$f(\mathbf{Z_t} \mid Z_{mt}) = (2\pi)^{-\frac{N}{2}} |\Sigma|^{-\frac{1}{2}} \times$$

$$\exp\left[-\frac{1}{2}(\mathbf{Z_t} - \alpha - \beta Z_{mt})' \Sigma^{\mathbf{-1}} (\mathbf{Z_t} - \alpha - \beta Z_{mt}) \right]$$

The Likelihood Function

- With T i.i.d. observations, the likelihood function is:

$$f(\mathbf{Z_1}, \mathbf{Z_2}, \ldots, \mathbf{Z_T} \mid Z_{m1}, \ldots, Z_{mT}) = \prod_{t=1}^{T} f(\mathbf{Z_t} \mid Z_{mt})$$

$$= \prod_{t=1}^{T} (2\pi)^{-\frac{N}{2}} |\Sigma|^{-\frac{1}{2}} \times$$

$$\exp\left[-\frac{1}{2} (\mathbf{Z_t} - \alpha - \beta Z_{mt})' \Sigma^{\mathbf{-1}} (\mathbf{Z_t} - \alpha - \beta Z_{mt}) \right]$$

MLE Estimates of Parameters

- Why do it this way? Because if you know the distribution, MLE's are
  - Consistent
  - Asymptotically efficient
  - Asymptotically normal
- In small samples, there may be problems. For instance, is the MLE of the mean unbiased?

The Log-Likelihood Function

- The log of the joint pdf viewed as a function of the unkown parameters, α, β, and Σ.

$$L(\alpha,\beta,\Sigma) = -\frac{NT}{2}\ln(2\pi) - \frac{T}{2}\ln|\Sigma|$$

$$-\frac{1}{2}\sum_{t=1}^{T}(\mathbf{Z_t} - \alpha - \beta Z_{mt})'\Sigma^{-1}(\mathbf{Z_t} - \alpha - \beta Z_{mt})$$

First-Order Conditions

- The ML parameter estimates maximize L. To find the estimators, set the FOCs to zero:

$$\frac{\partial L}{\partial \alpha} = \Sigma^{-1}\left[\sum_{t=1}^{T}(\mathbf{Z_t} - \alpha - \beta Z_{mt})\right]$$

- There are N of these derivataives one for each α_i.

FOCs Cont…

$$\frac{\partial L}{\partial \beta} = \Sigma^{-1}\left[\sum_{t=1}^{T}(\mathbf{Z_t} - \alpha - \beta Z_{mt})Z_{mt}\right]$$

- There are N of these as well, one for each β_i. Finally,

$$\frac{\partial L}{\partial \Sigma} = -\frac{T}{2}\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}\left[\sum_{t=1}^{T}(\mathbf{Z_t} - \boldsymbol{\alpha} - \boldsymbol{\beta} Z_{mt})(\mathbf{Z_t} - \boldsymbol{\alpha} - \boldsymbol{\beta} Z_{mt})'\right]\Sigma^{-1}$$

Solution

$$\hat{\alpha} = \hat{\mu} - \hat{\beta}\hat{\mu}_m$$

$$\hat{\beta} = \frac{\sum_{t=1}^{T}(\mathbf{Z_t} - \hat{\boldsymbol{\mu}})(Z_{mt} - \hat{\mu}_m)}{(Z_{mt} - \hat{\mu}_m)^2}$$

$$\Sigma = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{Z}_t - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}} Z_{mt})(\mathbf{Z}_t - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}} Z_{mt})' = \frac{1}{T}\sum_{t=1}^{T}\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\varepsilon}}'$$

where

$$\hat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{Z_t}, \quad \hat{\mu}_m = \frac{1}{T}\sum_{t=1}^{T}Z_{mt}$$

- These are just OLS parameters for α, and β.

Distributions of the Point Estimates

- The distributions of the MLE's conditional on the excess return of the market follows from the assumed joint normality of the excess returns and the i.i.d. assumption.
- The variances and covariances of the estimators can be derived using the Fisher Information Matrix.

The Fisher Information Matrix

- The information matrix is minus the matrix of second partials of the log-likelihood function with respect to the parameter vector.

$$-E\left[\frac{\partial^2 \ln(L)}{\partial\theta\,\partial\theta'}\right]$$

Then

$$\sqrt{T}(\hat{\theta}-\theta_0) \rightarrow N(0, I^{-1}(\theta_0))$$

$$\hat{I} = -\frac{1}{T}\sum_{t=1}^{T}\left[\frac{\partial^2 \ln L_t}{\partial\theta\,\partial\theta'}\right]$$

evaluated at the point estimates.

- The estimators are consistent and have the distributions:

$$\hat{\alpha} \sim N\left(\alpha, \frac{1}{T}\left[1+\frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]\Sigma\right)$$

$$\hat{\beta} \sim N\left(\beta, \frac{1}{T}\left[1+\frac{1}{\hat{\sigma}_m^2}\right]\Sigma\right)$$

$$T\hat{\Sigma} \sim W_N(T-2, \Sigma)$$

where

$$\hat{\sigma}_m^2 = \frac{1}{T}\sum_{t=1}^{T}(Z_{mt}-\hat{\mu}_m)^2$$

- $W_N(T\text{-}2, \Sigma)$ indicates that the NxN covariance matrix $T\Sigma$ has a Wishart distribution with T-2 degrees of freedom, a multivariate generalization of the chi-squared distribution.
- Note that $\hat{\Sigma}$ is independent of both $\hat{\alpha}$ *and* $\hat{\beta}$.

The Test Statistic

- We estimated the unconstrained market model to obtain the MLEs.
- Now, we impose the CAPM restrictions.
- If the CAPM is true, under the null:

$$H_0\text{: } \boldsymbol{\alpha} = \mathbf{0}$$

and under the alternative:

$$H_A\text{: } \boldsymbol{\alpha} \neq \mathbf{0}$$

- Having only estimated the unconstrained model, we employ the Wald test.

The Wald Test

- A straightforward application (see Greene or earlier notes).

$$J_0 = \hat{\boldsymbol{\alpha}}'\left[Var\left[\hat{\boldsymbol{\alpha}}\right]\right]^{-1}\hat{\boldsymbol{\alpha}}$$

which equals

$$T\left[1+\frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]^{-1}\hat{\boldsymbol{\alpha}}'\Sigma^{-1}\hat{\boldsymbol{\alpha}}$$

where we've substituted in for $Var[\hat{\alpha}]$.

Properties of J_0

- Under the null, $J_0 \sim \chi^2(N)$.
- Note that Σ is unknown.
- Substitute a consistent estimate of it into the statistic and then under the null the distribution is asymptotically chi-squared.
- The MLE of Σ is a consistent estimator.

We Can Do Better

- The Wald test is an asymptotic test.
- We, however, know the finite sample distribution.
- We can use this to do the Gibbons Ross and Shaken (1989) test.
- To do so, we will need the following theorem from Muirhead (1983).

- **Theorem:** Let the m-vector **x** be distributed N(**0**,**Ω**), let the (mxm) matrix **A** be distributed $W_m(n,\mathbf{\Omega})$ with n≥m, and let **x** and **A** be independent. Then:

$$\frac{(n-m+1)}{m}\mathbf{x}'\mathbf{A}^{-1}\mathbf{x} \sim F_{m,n-m+1}.$$

Applying the Theorem

- Let

$$\mathbf{x} = \left[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]^{-\frac{1}{2}} \hat{\boldsymbol{\alpha}},$$

$$\mathbf{A} = T\hat{\Sigma}$$

$$m = N, \quad n = T - 2$$

The GRS Test Statistic

$$J_1 = \left(\frac{T - N - 1}{N}\right)\left[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]^{-1} \hat{\boldsymbol{\alpha}}' \hat{\Sigma}^{-1} \hat{\boldsymbol{\alpha}}.$$

- Under the null, $J_1 \sim$ F(N,T-N-1).
- We can construct J_1 (and J_0) using only the estimators from the unconstrained model.

A Clever Interpretation of J_1

- GRS show that
$$J_1 = \left(\frac{T-N-1}{N}\right)\left(\frac{\frac{\hat{\mu}_q^2}{\hat{\sigma}_q^2} - \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}}{1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}}\right)$$
- q is the *ex-post* tangency portfolio constructed from the N assets plus the market portfolio.
- Recall that the portfolio with the maximum (squared) Sharpe ratio must be the tangency portfolio.
- When the *ex-post* q is m, $J_1 = 0$.
- As m's squared SR decreases, J_1 increases – evidence against the efficiency of m.

The Likelihood Ratio Test

- For the LR test, we must also estimate the constrained model, which is the S-L CAPM (α=0).
- FOCs:

$$\frac{\partial L}{\partial \beta} = \Sigma^{-1}\left[\sum_{t=1}^{T}(\mathbf{Z_t} - \boldsymbol{\beta} Z_{mt})Z_{mt}\right]$$

$$\frac{\partial L}{\partial \Sigma} = -\frac{T}{2}\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}\left[\sum_{t=1}^{T}(\mathbf{Z_t} - \boldsymbol{\beta} Z_{mt})(\mathbf{Z_t} - \boldsymbol{\beta} Z_{mt})'\right]\Sigma^{-1}$$

The Constrained Estimators

$$\hat{\boldsymbol{\beta}}^* = \frac{\sum_{t=1}^{T}(\mathbf{Z_t} Z_{mt})}{\sum_{t=1}^{T} Z_{mt}^2}$$

$$\hat{\Sigma}^* = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{Z_t} - \hat{\boldsymbol{\beta}}^* Z_{mt})(\mathbf{Z_t} - \hat{\boldsymbol{\beta}}^* Z_{mt})'.$$

- The estimators are consistent and have the following distributions (why T-1?):

$$\hat{\boldsymbol{\beta}}^* \sim N\left(\boldsymbol{\beta}, \frac{1}{T}\left[\frac{1}{\hat{\mu}_m^2 + \hat{\sigma}_m^2}\right]\Sigma\right)$$

$$T\hat{\Sigma}^* \sim W_N(T-1, \Sigma)$$

The LR Test

- Using a trick given in the text:

$$LR = L^* - L = -\frac{T}{2}\left[\ln|\hat{\Sigma}^*| - \ln|\hat{\Sigma}|\right]$$

- This test is based on the fact that –2 times the log of the likelihood ratio is asymptotically $\sim \chi^2$ with d.f. equal to the number of restrictions under the null.
- The test statistic is

$$J_2 = -2LR = T\left[\ln|\hat{\Sigma}^*| - \ln|\hat{\Sigma}|\right] \overset{a}{\sim} \chi_N^2$$

J_2 From J_1

- It turns out that we don't need large sample theory to conduct a LR test.
- The LR test statistic is a monotonic transformation of J_1 (see p. 195 of the text):

$$J_1 = \left(\frac{T-N-1}{N}\right)\left(\exp\left[\frac{J_2}{T}\right]-1\right)$$

Finite Sample LR Test

- Why:
  - Finite sample tests can differ from asymptotic tests by a lot, as we'll see when we discuss issues of size and power.
  - We'll see that the large sample tests reject "too often" in the case where we know the true small-sample distribution.
  - Call the adjusted large sample statistic J_3.

Jobson and Korkie (1982) Adjustment

$$J_3 = \left(\frac{T - N/2 - 2}{T} \right) J_2$$

which is also asymptotically distributed as a χ^2_N.

The Black CAPM

- Distinguishing feature: no risk-free asset.
- There must be a zero-beta CAPM, however given our assumptions.
- But, $E[R_{0m}]$ is now an unobservable and becomes something to estimate.
- There are also additional testable implications as explained in the text.

Black CAPM: Unconstrained Model

$$\mathbf{R_t} = \boldsymbol{\alpha} + \boldsymbol{\beta} R_{mt} + \boldsymbol{\varepsilon}_t$$

$$E[\boldsymbol{\varepsilon}_t] = \mathbf{0}$$

$$E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'] = \Sigma$$

$$E[R_{mt}] = \mu_m \quad E[(R_{mt} - \mu_m)^2] = \sigma_m^2$$

$$Cov[R_{mt}, \varepsilon_t] = 0$$

Testable Implication

$$\boldsymbol{\alpha} = (\mathbf{1} - \boldsymbol{\beta})\boldsymbol{\gamma}$$

- This is more complicated to test than the S-L CAPM because β and γ enter in a non-linear fashion.
- Still, with i.i.d. jointly normal returns, we can use the ML approach.
- This is the contribution of Gibbons' (1982) dissertation.
- You can see the solution in the text.

Size and Power

- We want to examine the appropriateness of using large-sample theory when sample size is not large.
- We will investigate the appropriateness of the large sample tests when:
  - Returns are multivariate normal.
  - This means that the multivariate F-test is the appropriate test and J_1 is the correct test statistic.
  - The question is, what happens if you use one of the large sample test statistics we derived earlier?

Comparison

- The Wald test, J_0 vs J_1: $$J_1 = \frac{T-N-1}{NT} J_0$$
- The LR test, J_2 vs J_1:

$$J_1 = \frac{T-N-1}{N}\left(\exp\left[\frac{J_2}{T}\right]-1\right)$$

- The Jobson and Korkie LR test, J_3 vs J_1:

$$J_1 = \frac{T-N-1}{N}\left(\exp\left[\frac{J_3}{T-N/2-2}\right]-1\right)$$

Size of the Tests

- $J_1 \sim F(N, T - N - 1)$.
- J_0, J_2, J_3 are all asymptotically $\chi^2(N)$.
- Calculate the exact size of each asymptotic test based on its asymptotic 5% critical value.
- e.g. with 10 portfolios and 60 months of data, $J_0 \sim \chi^2(10)$.
- So, the critical value for a test with asymptotic size of 5% is 18.31.
- We know how J_0 and J_1 are related. Convert 18.31 into the corresponding J_1. This is 1.495.

Size of the Tests Cont…

- $J_1 = 1.495$.
- And, $J_1 \sim F(N, T - N - 1) = F(10,49)$.
- This corresponds to a size of 17.0%.
- Thus an asymptotic 5% test has a size of 17% in a sample of 60 months.
- This means, if you use the asymptotic test, you'll reject the null (wrongly) more than three times too often!

Interpretation

- The Table on pg. 205 of CLM illustrates this.
  - The problem is worse for a small sample.
  - It is particularly bad for the Wald test.
  - Fixing T, as N grows large, the problem becomes worse.
  - The Jobson and Korkie adjustment works very well.
- So asymptotically really means asymptotic. Finite sample adjustments can be very important.

Power

- Power: The probability of rejecting the null when the alternative is true.
  - Low power against an interesting alternative suggests that the test has no useful discriminating ability.
    - Power is always computed against a particular alternative.
  - With low power, failing to reject the null is not indicative of much.

Example Featuring J_1

- Reason to use J_1: exact finite-sample distribution is known under both the null and alternative hypotheses.
- Conditional on the realized excess return of the market portfolio $J_1 \sim F_{N,T-N-1}(\delta)$, where δ is the non-centrality parameter of the F distribution:

$$\delta = T\left[1+\frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]^{-1} \boldsymbol{\alpha}' \Sigma^{-1} \boldsymbol{\alpha}$$

- To specify the distribution of J_1 under both the null and the alternative, we need δ, N, and T.

Example Featuring J_1 Cont…

- Under the null, the market portfolio is efficient (it has the maximal Sharpe Ratio).
- Therefore, $\alpha = \mathbf{0}$ and J_1 is a central $F_{N,T-N-1}$.
- We need an alternative. Economically, this means that we need another value for the Sharpe ratio of the market portfolio.
- Given that value, we can compute δ, the χ^2 non-centrality parameter under the alternative.

Example Featuring J_1 Cont…

- This means we need
  - $\hat{\mu}_m^2/\hat{\sigma}_m^2$, the squared Sharpe ratio for the market portfolio as well as α and Σ. Or we could use:
  - $\alpha'\Sigma^{-1}\alpha$
- For $\hat{\mu}_m^2/\hat{\sigma}_m^2$, choose 0.013, which corresponds to an *ex-post* annualized excess return of 8% with a s.d. of 20%.
- For $\alpha'\Sigma^{-1}\alpha$, use the GRS result:

$$\boldsymbol{\alpha}'\Sigma^{-1}\boldsymbol{\alpha} = \frac{\hat{\mu}_q^2}{\hat{\sigma}_q^2} - \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2} = sr_q^2 - sr_m^2.$$

Example Featuring J_1 Cont…

- So we need the S.R.'s for the market portfolio and the tangency portfolio.
- What we'll do is make up the Sharpe ratio for the tangency portfolio.
- Let the s.d. of the tangency be 16%.
- Let the mean be one of four values: 8.5%, 10.2%, 11.6%, and 13%.

Table

- This corresponds to δ/T's of .01, .02, .03, and .04.
- In addition, specify N = 1, 5, 10, 20, and 40.
- Let T = 60, 120, 240, and 360.
- Let the size of the test be 5%.
- Then, under the null, the Sharpe ratio of the market portfolio is equal to that of the tangency portfolio.
- Under the alternative, the squared Sharpe ratio equals 0.013.
- Then, the table (pg. 207) gives the corresponding power of the test.

Comments

- There is substantial variation in power.
  - Fix N. Increasing T improves power.
    - Any issues?
  - Fix T. Reducing N increases power. Unfortunately, the S-R of the tangency portfolio decreases as N decreases.
    - Careful choice of the right "assets" is important.
  - According to CLM, keep N small, perhaps no larger than 10.