Intertemporal Equilibrium Models
CLM Chapter 8

The Stochastic Discount Factor

- The mother of all asset pricing models
- Consider an investor who can trade freely in asset i, and maximizes the expectation of a time separable utility function.

$$Max\ E_t\left[\sum_{j=0}^{\infty}\delta^j U(C_{t+j})\right]$$

- where C is consumption, and δ is the time discount factor.

First order conditions

$$U'(C_t) = \delta E_t\left[R_{it+1}U'(C_{t+1})\right]$$

- where $R_{it+1}=(1+r_{it+1})$
- Divide both sides by $U'(C_t)$

$$1 = E_t\left[R_{it+1}M_{t+1}\right]$$

- where $M_{t+1} = \delta U'(C_{t+1})/U'(C_t)$
- The variable M_{t+1} is known as the *stochastic discount factor* or *pricing kernel.*

Intuition

- Working with the unconditional form we can write

$$E[R_{it}] = \frac{1}{E[M_t]} - \frac{Cov(R_{it}, M_t)}{E[M_t]}$$

- Given that E[M] is positive (show this), assets that have a low covariance with M have higher expected returns (Also show that $R_f=1/E[M]$)
  - M is high when the investor has a high marginal utility of future consumption
  - A low covariance implies that the asset has low returns when utility of consumption is high--Risky.

Generalizing

- The equation can also be derived only by assuming the absence of arbitrage (see CLM page 295)
- This is important because it says that all arbitrage free asset pricing models can be written in this form
- Example: Assume that

$$M_t = a + bR_{mt}$$

Volatility Bounds (Hansen and Jaganathan, 1991)

- Using asset returns we can learn about the behavior of the SDF. HJ show how to derive a lower bound on the variability of the set of permissible discount factors that are consistent with observed asset returns. Using excess returns

$$0 = E(R_i^e M) = E[M]E[R_i^e] + Cov(R_i^e, M)$$

- Use the fact that $Cov(A,B) = \rho(A,B)\sigma(A)\sigma(B)$

Volatility Bounds

- This implies that

$$\sigma(M) = \frac{-E[M]E[R_i^e]}{\rho(R_i^e, M)\sigma(R_i^e)}$$

- And therefore, that

$$\sigma(M) \geq \frac{E[M]E[R_i^e]}{\sigma(R_i^e)} = \frac{1}{R_f}\left[\frac{E[R_i^e]}{\sigma(R_i^e)}\right]$$

Equity Premium Puzzle

- This equation places a lower bound on the variability of any stochastic discount factor that can possibly be consistent with the observed asset returns. This is essentially the basis of the "Equity Premium Puzzle" of Mehra and Prescott (1985)
- Assume that investors have power utility:

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

Equity Premium Puzzle

- Then we can write the pricing equation as:

$$1 = E_t\left[R_{it+1}\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]$$

- Assuming that asset returns and aggregate consumption are jointly lognormal (can be relaxed using GMM) it can be shown that:

$$E_t[r_{it+1} - r_{ft+1}] + \frac{\sigma_i^2}{2} = \gamma\sigma_{ic}$$

Sample Moments (1889-1994)

Variable	Mean	Std. Dev	Corr. w/cons. growth	Covariance w/cons. growth
Cons. Growth	.0172	.0328	1.000	.0011
Stock returns	.0601	.1674	.04902	.0027
CP-returns	.0183	.0544	-.1157	-.0002
Excess returns	.0418	.1774	.4979	.0029

Equity Premium Puzzle

- Using the data in the table, and the pricing equation we can back out the implied coefficient of risk aversion:

$$\gamma = \frac{0.0418 + (0.1674)^2 / 2}{0.0029} = 19.2$$

- This is very high (too high to be reasonable).

Equity Premium Puzzle

- Another way to frame the puzzle is using the HJ bounds. The stock return data imply that any candidate SDF must have a standard deviation of 0.33. Using a log approximation of the SDF:

$$Var(M_{t+1}) \approx Var(m_{t+1}) = \gamma^2 Var(\Delta c_{t+1})$$

- Which implies γ=10 if stock returns are perfectly correlated with consumption. Using the empirical correlation of about 0.5 implies γ=20

Equity Premium Puzzle

- The underlying issue is that stock returns are much more variable than consumption growth and have low correlations with consumption growth.
- To attempt to solve the equity premium puzzle one path has been to use alternative utility functions (e.g., habit formation).
- Another is to use conditional versions of asset pricing models (e.g., the WFC paper).