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Working Orders in Limit Order Markets and Floor Exchanges

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ABSTRACT

We analyze limit order markets and floor exchanges, assuming an informed trader and discretionary liquidity traders use market orders and can either submit block orders or work their demands as a series of small orders. By working their demands, large market order traders pool with small traders. We show that every equilibrium on a floor exchange must involve at least partial pooling. Moreover, there is always a fully pooling (worked order) equilibrium on a floor exchange that is equivalent to a block order equilibrium in a limit order market.

THE GOAL OF THIS PAPER IS TO COMPARE ALTERNATIVE MARKET designs. Glosten (1994) shows that a market with an open limit order book is robust to competition from other markets. Our principal result is that other market types may mimic an open limit order book and hence have the same robustness. In particular, following Glosten (1994) and assuming perfect competition among risk-neutral liquidity providers, a uniform price auction has an equilibrium that is equivalent in all important respects to the equilibrium of an open limit order book. We argue below that uniform pricing is an essential feature of a floor exchange, and thus an equivalence between (stylized models of) limit order markets and floor exchanges obtains. In these stylized models, the distinction between limit order markets and floor exchanges is that pricing in a limit order market is discriminatory (each limit order executes at its limit price rather than the marginal price) whereas pricing on a floor exchange is uniform (all shares in a trade execute at the same price). Note that we do not model other important distinctions between limit order markets and floor exchanges, in particular, the anonymity of orders (and hence the potential for building reputations) and the extent to which orders are revealed (and hence the opportunity for front running).[1]

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[1] See Benveniste, Marcus, and Wilhelm (1992) for an analysis of the role of reputation on a floor exchange.

A market with an open limit order book can be viewed as a screening game. In contrast, a floor exchange, in which floor traders compete to fill an order after observing its size, can be viewed as a signaling game. Consistent with the fact that signaling games tend to have many equilibria, we show that floor exchanges can have equilibria that are not equilibria of an open limit order book. However, every equilibrium of a floor exchange must involve at least partial pooling of large traders with small traders.[2]

Large traders can pool with small traders in our model because we allow traders to work orders, that is, to execute a large order as a series of small orders. This is the primary innovation of our paper. To do this requires a dynamic model, and, as far as we know, this is the first paper to compare alternative market designs using a dynamic model. Our model is a generalization of Back-Baruch (2004), which is essentially a continuous-time Glosten-Milgrom (1985) model with optimal trading by a single informed trader (one could also say that it is a continuous-time Kyle (1985) model with discrete order sizes and Poisson arrival of liquidity trades). The key condition we use from Back-Baruch (2004) is that the informed trader is always willing to trade (he plays a mixed strategy, randomizing between trading and waiting). We use this first-order condition to compare the costs to liquidity traders of either (i) submitting block orders or (ii) working orders. We assume liquidity traders have discretion to choose the cheapest way to trade. We allow traders to work orders by submitting a series of orders an instant apart, achieving execution at essentially the same time as if they placed a block order. We show that it is never an equilibrium in a floor exchange for all traders to use block orders (there must be at least partial pooling). Moreover, the block order equilibrium in a limit order market, which appears to always exist, is equivalent to a worked order equilibrium in a floor exchange. These equilibria (specifically the block order equilibrium in a limit order market) are the equilibria shown by Glosten (1994) to be "inevitable."

Ask prices in limit order markets with risk-neutral competitive liquidity providers are "upper-tail expectations"; similarly, bid prices are "lower-tail expectations." In contrast, prices in uniform price auctions are expected values conditional on order size. As already mentioned, and as will be discussed further below, we believe this is a reasonable model of a floor exchange. Assuming that ask prices are expectations on a floor exchange and upper-tail expectations in a limit order market, it follows that prices for small orders "should" be better on a floor exchange than in a limit order market. This is well known (see, for example, Glosten (1994) or Seppi (1997)). The "should" here presumes a separating equilibrium whereby small orders, which have less information content, can be distinguished from large orders. However, it is precisely the favorable prices for small orders on a floor exchange when traders are separated (submit block orders) that cause large traders to want to pool with small traders by working orders, that is, to deviate from the hypothetical separating equilibrium.

[2] More precisely, we show this is true of each Markov equilibrium in which the informed trader's value function is monotone. We assume the monotonicity and Markovian properties in each of our results, but we do not repeat this caveat continually. See footnote 12 for some additional discussion.

When orders are worked, liquidity providers on a floor exchange can of course condition on the size of an order, but they cannot condition on the size of the demand underlying the order. In other words, they cannot know whether more orders from the same trader in the same direction will be immediately forthcoming. Thus, in a pooling equilibrium on a floor exchange, ask prices are upper-tail expectations—expectations conditional on the size of the demand being the size of the order or larger—precisely as in a limit order market. This is the reason a pooling (worked order) equilibrium on a floor exchange is equivalent to a block order equilibrium in a limit order market.

Floor exchanges are organized in a variety of ways. Any floor exchange has numerous rules that are not precisely captured by our model. However, the essence of such an exchange is exposing each market order to the trading crowd so that competition among the members of the crowd generates the best available price for the order.[3] We assume the trading crowd consists of two or more risk-neutral liquidity providers who maximize expected trading profits. Bertrand competition among the liquidity providers implies that a market order's price is the expectation of the asset value conditional on past information and on the size of the order. Examples of a trading crowd would be market makers on the Chicago Board of Exchange or floor brokers engaging in proprietary trading on the New York Stock Exchange. Of course, the risk neutrality assumption may be counterfactual, but the assumption that the trading crowd competes to fill orders after observing the size of each order seems reasonable.

The result that fully pooling (worked order) equilibria exist and mimic the block order equilibrium in a limit order market applies to the following:[4] (i) floor exchanges with risk-neutral competitive floor traders, (ii) hybrid limit order book/floor exchanges in which limit order traders are risk neutral and competitive and floor traders are risk neutral and competitive, and (iii) hybrid limit order book/floor exchanges in which limit order traders are risk neutral and competitive and the floor consists of a monopolist specialist. In the latter two cases, floor traders impose adverse selection on limit order traders by being able to condition on order size. In case (ii), pricing is exactly the same as in a uniform price auction and our results apply directly. In case (iii)—the case analyzed by Rock (1990)—it is again true that there is a worked order equilibrium that is equivalent to a block order equilibrium in a limit order market. The intuition is that the informational advantage of floor traders analyzed by Rock (1990), and hence the adverse selection imposed on limit order traders, disappears when all orders are worked because in that case neither limit order traders nor floor traders can condition on the size of the demand underlying a market order.

[3] For example, NYSE rules state that market orders request execution "at the most advantageous price obtainable after the order is represented in the Trading Crowd" (cited by Hasbrouck, Sofianos, and Sosebee (1993, p. 4)).

[4] A previous version of this paper showed that the result also applies to a proposal (subsequently dropped) by the NYSE to impose uniform pricing on any part of a market order that exceeds the depth at the inside quote (by price-improving all executed limit orders other than those at the inside quote to the marginal limit price).

Most of the literature on market microstructure assumes uniform pricing, as we do in our model of a floor exchange. For example, the Kyle (1985) model and all of its variations assume uniform pricing. Most of the literature that applies the Glosten-Milgrom (1985) framework assumes single-unit demands, so there is no distinction between uniform pricing and discriminatory pricing ("discriminatory" means that each limit order executes at its stated price, rather than at the marginal price). Uniform pricing is also assumed by two notable papers that do consider multiple order sizes in a Glosten-Milgrom framework, namely, Easley and O'Hara (1987) and Seppi (1990).

We are not the first to conclude that orders should be worked in uniform price markets. For example, in Kyle's (1985) dynamic model, the informed trader trades gradually. More closely related to our work is Chordia and Subrahmanyam (2004), who show in a two-period model with normally distributed liquidity demands and informed trading in the second period that discretionary liquidity traders should split their orders between the two periods. The contribution of our paper is to analyze the working of orders in limit order markets and to draw the connection between worked order equilibria in uniform price markets and block order equilibria in limit order markets.

Important theoretical papers on limit order markets, assuming discriminatory pricing, include the following:

- Rock (1990) describes the adverse selection imposed by floor traders on limit order traders.
- Glosten (1994) demonstrates the robustness of limit order markets vis-a-vis competition from other markets, assuming perfect competition in liquidity provision.
- Bernhardt and Hughson (1997) show that strategic competition among a finite number of liquidity providers using limit orders must result in positive profits for the liquidity providers.
- Seppi (1997) compares a specialist market (in which the specialist faces competition in liquidity provision from floor traders) to a limit order market, assuming perfect competition in liquidity provision.
- Ready (1999) extends Rock's (1990) analysis of the adverse selection imposed by a specialist on a limit order book, assuming a single order size. His model is dynamic, but limit order traders move first and cannot cancel orders that are unexecuted.[5]
- Biais, Martimort, and Rochet (2000) extend Bernhardt and Hughson (1997), providing additional analysis of strategic competition among liquidity providers in limit order markets.
- Viswanathan and Wang (2002) compare limit order markets and uniform price markets, assuming strategic competition among liquidity providers.

[5] In contrast, we assume limit order traders continuously monitor the market, canceling and resubmitting orders instantaneously. The truth obviously lies somewhere between. Interesting evidence on this point is provided by Hasbrouck and Saar (2002), who show that more than one-fourth of the limit orders on the Island ECN are canceled within two seconds or less, though at least some such orders may be from liquidity demanders rather than liquidity providers.

In their model of a uniform price market, a finite number of “dealers” submit demand-supply schedules before the size of the market order is known.
- Parlour and Seppi (2003) extend the analysis of Seppi (1997) and analyze competition among exchanges.
- Glosten (2003) compares limit order markets and uniform price markets, assuming perfect competition in liquidity provision. He endows market order traders with preferences and derives an equilibrium with optimizing market order traders as well as optimizing liquidity providers.

Of these papers, the ones that are most closely related to this paper are Seppi (1997), Viswanathan and Wang (2002), and Glosten (2003). Each of these compares limit order markets to uniform price markets. The key distinction between their analyses and the analysis in this paper is that we endogenize informed trading (and, to a certain extent, uninformed trading) in a dynamic model. Seppi (1997) and Viswanathan and Wang (2002) analyze the market structures at a point in time, taking the market order flow as given and assuming it is the same in both types of markets. Glosten (2003) points out that the market order flow should depend on the market structure, and he endogenizes it but still within a static model. Thus, the previous literature does not capture the option of working large orders as a series of small orders. As we mentioned above, modeling this option is the primary innovation of our paper.

We do not address the optimality of using market orders versus limit orders for traders that are motivated to trade due to informational reasons or liquidity reasons. Papers that address the choice between market orders and limit orders include Kumar and Seppi (1993), Chakravarty and Holden (1995), Handa and Schwartz (1996), Harris (1998), Parlour (1998), Foucault (1999), Foucault, Kadan, and Kandel (2005), Goettler, Parlour, and Rajan (2004, 2005) and Rosu (2005). We also do not analyze competition among exchanges; see, for example, Glosten (1994), Parlour and Seppi (2003), Hendershott and Mendelson (2000), and Viswanathan and Wang (2002) for papers that consider this issue.

The plan of our paper is as follows. Sections I to III describe the two types of markets assuming traders all submit block orders (Section I describes the elements of the model that are common to the two market types, Section II describes limit order markets, and Section III describes uniform price markets). Sections IV and V show that there are block order equilibria in limit order markets but no equilibria with exclusively block orders in uniform price markets. Section VI describes equilibria in uniform price markets in which traders work orders by submitting orders instantaneously one after the other. We show in Section VI that an equilibrium with worked orders in a uniform price market is equivalent to a block order equilibrium in a limit order market and that there may be equilibria in a uniform price market in which some orders are worked and some are submitted as blocks. Section VII discusses hybrid markets consisting of a floor and a limit order book, and Section VIII concludes.

I. Basic Model

Our model, a generalization of the model of Back and Baruch (2004), is perhaps the simplest possible model of endogenous informed trading with multiple order sizes.[6] We consider a continuous-time market for a risky asset and one risk-free asset with interest rate set to zero.[7] There is no minimum tick size—any real number is a feasible transaction price—and there are either an infinite number of risk-neutral limit order traders, or, in a uniform price market, at least two risk-neutral traders who compete in a Bertrand fashion to fill incoming market orders. Market orders are submitted by a single informed trader and by "liquidity" traders. A public release of information takes place at a random time τ, distributed as an exponential random variable with parameter r. After the public announcement has been made, the value of the risky asset, denoted by $\tilde{v}$, will be either zero or one, and all positions are then liquidated at that price.[8] All trades are anonymous. The single informed trader knows $\tilde{v}$ at date 0. If $\tilde{v} = 1$ we say that the informed trader is the "high" type, and if $\tilde{v} = 0$ we say he is the "low" type.

We consider orders of size i for $i = 1, \ldots, n$, where n is an arbitrary but fixed integer. We assume buy and sell orders by liquidity traders are Poisson processes with constant arrival intensities. For simplicity, the arrival intensities for buy and sell orders by liquidity traders are assumed to be the same. For orders of size i, the arrival intensity of buys and sells is denoted by β_i.

There are three parts to our definition of equilibrium:

(i) The informed trader maximizes expected profits.
(ii) Limit prices are tail expectations and uniform prices are expectations, conditional on the history of orders.
(iii) Liquidity traders choose the method of trading—block orders or worked orders—that provides the best execution.

Regarding part (i), our results rely upon first-order conditions for expected profit maximization and monotonicity of value functions (conditions (6), (7), and (11) below). In part (iii), the discretion we allow liquidity traders is that they can work their demands by submitting a series of orders an instant apart, thereby achieving execution at essentially the same time as if they submitted

[6] We generalize Back and Baruch (2004) by allowing multiple order sizes and by studying limit order markets in addition to uniform price markets.

[7] Obviously, a one-period model would be simpler, but such a model cannot capture the ability of a trader to transact a large quantity by submitting a series of small orders. This is an important issue in the choice of order size, and it requires a dynamic model. The standard dynamic models (Kyle (1985), Glosten and Milgrom (1985)) are inadequate for our purposes, the Kyle model because it imposes a uniform price market, and the Glosten-Milgrom model because it does not endogenously determine who trades at each date.

[8] One can also think of the announcement date τ as a random time at which one or more traders other than the single informed trader in the model learn the information $\tilde{v}$ and it becomes common knowledge that this is the case. As Holden and Subrahmanyam (1992) and Back, Cao, and Willard (2000) discuss, competition between identically informed risk-neutral traders will push the asset price immediately to $\tilde{v}$.

a block order. Note that, until Section IV, we assume that liquidity traders (and hence the informed trader) submit block orders, ignoring part (iii) of the definition of equilibrium.

We look for equilibria in each market type in which the conditional expectation of the asset value, given the information of liquidity providers, is a Markov process. We let m_t denote the conditional expectation at date t. Of course, because the asset value is either zero or one, m_t also denotes the conditional probability that the asset value is one. We define $m_{t-} = \lim_{s \uparrow t} m_s$, which can be interpreted as the conditional expectation just before observing whether an order is submitted at date t. The conditional expectation m_{t-} will be the state variable for the Bayesian updating of liquidity providers and the optimization of the informed trader at date t. We assume that liquidity providers are uncertain about the asset value at the initial date ($0 < m_0 < 1$), but the precise value of m_0 is irrelevant for our results.

Let $a_i(m)(b_i(m))$ denote the conditional expectation of the asset given a buy (sell) order of size i at date t when $m_{t-} = m$. Let $a_{i+}(m)(b_{i+}(m))$ denote the expectation conditional on a buy (sell) order of size i or greater. We will verify that $a_j(m) > a_i(m)$ and $b_j(m) < b_i(m)$ for all $j > i$ and all $m \in (0, 1)$; thus, larger orders have more information content in equilibrium. This implies that $a_{i+}(m) > a_i(m)$ and $b_{i+}(m) < b_i(m)$ for each order size i. Of course, because the conditional expectations depend on the trading strategies, they will vary across the two market types. When necessary for clarity, we use superscripts L and U to denote the limit order market and uniform price market, respectively.

Focusing on the buy side, as the sell side is symmetric, our assumption about liquidity provision (part (ii) of the definition of equilibrium) implies that at a point in time pricing in each of the two market types is as follows.

Limit Order Market: For each i, there will be a limit sell order for one unit at price $a^L_{i+}(m_{t-})$. The cost of a buy order of size i will be $\sum_{j=1}^{i} a^L_{j+}(m_{t-})$.

Uniform Price Market: The cost of a buy order of size i will be $ia^U_i(m_{t-})$.

Competition among limit order traders enforces a zero expected profit condition, implying that the book is as described. The inside ask must be the upper-tail expectation a^L_{1+}, because the inside limit sell order will transact against all market buy orders. Likewise, the other limit ask prices must be their corresponding upper-tail expectations.

We need to introduce notation for the stochastic processes that count the number of orders of each type.[9] The counting processes for liquidity trades are denoted by Z, the counting processes for informed trades by X, and the counting process for total trades by Y ($Y = X + Z$). We use superscripts $+$ and $-$ to denote counting processes for buy and sell orders, respectively. For example, Z^+_{it} denotes the total number of buy orders of size i by liquidity traders through time t. It jumps up by one each time a liquidity buy order of size i arrives. We are

[9] The equilibrium order processes and the conditional expectation process m will in general depend on the market type, even though our notation does not indicate it.

assuming that Z_i^+ and Z_i^- are Poisson processes with intensities β_i. Likewise, X_i^+ counts the buy orders of size i from the informed trader, and $Y_i^+ = X_i^+ + Z_i^+$ counts the total buy orders of size i.

Defining $X_i = X_i^+ - X_i^-$ and $Y_i = Y_i^+ - Y_i^-$, the process Y_i jumps up by one when any buy order of size i arrives and jumps down by one when any sell order of size i arrives. The information of liquidity providers is given by the vector process $Y = (Y_1, \ldots, Y_n)$. Without loss of generality (given risk neutrality), we assume the informed trader has no initial position in the risky asset, so $\sum_{i=1}^n iX_{it}$ denotes the number of shares the informed trader owns at date t.

The informed trader must play a mixed strategy in equilibrium. To see this, suppose to the contrary that the equilibrium strategies call for some amount i of the asset to be purchased by the high type but not the low type at some date t which is either fixed or a stopping time measurable with respect to Y. Then, the prior probability assessed by liquidity providers of this purchase at date t being made by the high type is m_{t-}, and the prior probability of it being made by a liquidity trader is $\beta_i\,dt$. Bayes's rule, therefore, implies that the updated expectation of the asset value upon observing the purchase is $a_i(m_{t-}) = 1$. This means that the high type would not profit from the purchase; furthermore, the purchase would eliminate all future profit opportunities. On the other hand, if the high type refrains from this purchase that the liquidity providers anticipate him making, then they will infer he must be the low type, implying $m_t = 0$ and therefore that the high type can make infinite future profits. Clearly, a trade at a known date t (knowable from the history of trades) cannot be an equilibrium strategy.

Playing a mixed strategy means that the informed trader mimics the liquidity traders by trading at random dates: His counting processes X_i^+ and X_i^- have arrival intensities analogous to the Poisson arrival intensities of the liquidity traders (though, unlike a Poisson process, the arrival intensities of the informed trader need not be constants). We therefore look for equilibria in which there exist functions θ_i^+ and θ_i^- (that will depend on the market type) such that for each order size i the stochastic processes

$$X_{it}^+ - \int_0^t \theta_i^+(m_{s-}, \tilde{v})\,ds \tag{1a}$$

$$X_{it}^- - \int_0^t \theta_i^-(m_{s-}, \tilde{v})\,ds \tag{1b}$$

are martingales relative to the informed trader's information. For example, when $\tilde{v} = 1$,

$$\text{prob}_{t-}\big(dX_{it}^+ = 1\big) = E_{t-}\big[dX_{it}^+\big] = \theta_i^+(m_{t-}, 1)\,dt, \tag{2}$$

so $\theta_i^+(m_{t-}, 1)$ denotes the probability with which the high-type informed trader submits a buy order of size i, per unit of time.

Some of the arrival intensities can be zero. However, for each order size used by liquidity traders, the high-type trader must submit a buy order of that size with positive probability and the low-type trader must submit a sell order of that size with positive probability at each instant. Otherwise, a profitable order will not affect beliefs adversely and hence should be used with probability one, contradicting the assumption that it is used with zero probability.

We can now characterize the evolution of the conditional expectation m_t over time. It jumps up or down when an order arrives and may also evolve between transactions.[10] We can write its dynamics in either market type (omitting the superscripts L and U) as

$$dm_t = f(m_{t-})\,dt + \sum_{i=1}^{n}[a_i(m_{t-}) - m_{t-}]\,dY_{it}^{+} + \sum_{i=1}^{n}[b_i(m_{t-}) - m_{t-}]\,dY_{it}^{-}. \quad (3)$$

Equation (3) means that the conditional expectation jumps up to a_i when there is a buy order of size i ($dY_{it}^{+} = 1$) and jumps down to b_i when there is a sell order of size i ($dY_{it}^{-} = 1$), and between transactions it evolves as $dm_t = f(m_{t-})\,dt$, where f is a function that is to be determined. We take zero and one to be absorbing points for m, because further information cannot change beliefs that put probability one on the asset value being low or probability one on the asset value being high. *Note that everything in equation (3)—the equilibrium aggregate order process Y; the functions f, a_i, and b_i; and the equilibrium conditional expectation process m—depends in general on the market type.*

II. Limit Order Markets

In this section, we present some basic facts about our model of limit order markets. We assume throughout the section that liquidity traders submit block orders, that is, we ignore part (iii) of the definition of equilibrium.

Consider the informed trader's optimization problem. The profit earned by the informed trader on a buy order of size i at date t is

$$i\tilde{v} - \sum_{j=1}^{i} a_{j+}(m_{t-}). \quad (4a)$$

The summation represents walking up the book. Likewise, the profit earned on a sell order of size i is

$$\sum_{j=1}^{i} b_{j+}(m_{t-}) - i\tilde{v}. \quad (4b)$$

The informed trader chooses a trading strategy $X = (X_1, \ldots, X_n)$ to maximize his expected cumulative profits until the announcement date τ:

[10] The conditional expectation should change between transactions because informed traders with different information will trade with different intensities. The absence of a trade indicates that the information of the trader is more likely to be consistent with a low intensity of trading than with a high intensity. Diamond and Verrecchia (1987) and Easley and O'Hara (1992) obtain a similar result, though in different models.

$$E\int_0^{\tau}\left\{\sum_{i=1}^{n}\left(i\tilde{v}-\sum_{j=1}^{i}a_{j+}(m_{t-})\right)dX_{it}^{+}+\sum_{i=1}^{n}\left(\sum_{j=1}^{i}b_{j+}(m_{t-})-i\tilde{v}\right)dX_{it}^{-}\right\}. \quad (5)$$

The informed trader computes this expectation knowing the value $\tilde{v}$ of the asset. Integrating with respect to the counting processes X_i^+ and X_i^- simply adds up the profit earned at each date a buy or sell order of size i is submitted, that is, the profit at the dates t when $dX_{it}^+ = 1$ or $dX_{it}^- = 1$. The informed trader chooses a trading strategy to maximize the expectation (5) conditional on $\tilde{v}$ and subject to the dynamics (3) for m, where in (3) he takes f and the a_i and b_i to be exogenously given functions and in (5) he takes the a_{j+} and b_{j+} to be exogenously given functions.

In this maximization problem, we allow the informed trader to choose arbitrary counting processes. However, we search for a mixed strategy equilibrium as defined in (1) in which $\theta_i^+(m, 1) > 0$ and $\theta_i^-(m, 0) > 0$ for each $m \in (0, 1)$ and each order size i. This means that the high type randomizes over buying in all possible sizes and the low type randomizes over selling in all possible sizes. The first-order conditions for such a strategy to be optimal are straightforward. Because the objective function (5) is stationary, we can define the value at any date t as a function of the state variable m_{t-} and the asset value v. Let $J(m, v)$ denote the value function. An equilibrium in which the high type submits buy orders of all sizes with positive probabilities and the low type submits sell orders of all sizes with positive probabilities must satisfy the following conditions for each $m \in (0, 1)$ and each order size i:

$$J(m,1) = i - \sum_{j=1}^{i} a_{j+}(m) + J(a_i(m), 1), \quad (6a)$$

$$J(m,1) \geq \sum_{j=1}^{i} b_{j+}(m) - i + J(b_i(m), 1), \text{ with equality when } \theta_i^-(m,1) > 0, \quad (6b)$$

$$J(m,0) = \sum_{j=1}^{i} b_{j+}(m) + J(b_i(m), 0), \quad (6c)$$

$$J(m,0) \geq -\sum_{j=1}^{i} a_{j+}(m) + J(a_i(m), 0), \text{ with equality when } \theta_i^+(m,0) > 0. \quad (6d)$$

Condition (6a) means that the optimal value for the high type can be realized by submitting a buy order of size i. The effect of submitting such an order is an instantaneous profit[11] and a continuation value of $J(a_i(m), 1)$. Condition (6b) means that submitting a sell order of size i is not a strictly superior strategy for the high type but it must be an optimal strategy when such an order is

[11] The profit is actually realized at the announcement date τ, but we normalize the interest rate to zero.

submitted with positive probability. Conditions (6c) and (6d) have analogous interpretations for the low type.

To have an equilibrium in which the informed trader buys or sells at random times, it must also be optimal for the informed trader to refrain from trading at any point in time. If he does not trade, then during an instant dt the announcement will occur with probability $r\,dt$, and if the announcement occurs the value function becomes 0. An uninformed buy order of size i will arrive with probability $\beta_i\,dt$, in which case m will jump to $a_i(m_{t-})$ and the value function will jump (up or down) to $J(a_i(m_{t-}), \tilde{v})$. Similarly, with probability $\beta_i\,dt$ an uninformed sell order of size i will arrive and the value function will jump to $J(b_i(m_{t-}), \tilde{v})$. Finally, in the absence of an announcement or an order, m will change by $f(m_{t-})\,dt$ and the value function will change by

$$\frac{\partial J(m_{t-}, \tilde{v})}{\partial m} f(m_{t-})\,dt.$$

For the informed trader to optimally refrain from trading, all of these expected changes in the value function must cancel, which means that, for each $m = m_{t-} \in (0, 1)$ and each $v \in \{0, 1\}$, we must have

$$rJ(m, v) = \frac{\partial J(m, v)}{\partial m} f(m) + \sum_{i=1}^{n} \beta_i [J(a_i(m), v) - J(m, v)] + \sum_{i=1}^{n} \beta_i [J(b_i(m), v) - J(m, v)]. \tag{7a}$$

The natural monotonicity and boundary conditions are, for all $m < m'$,

$$0 = J(0, 0) < J(m, 0) < J(m', 0) < J(1, 0) = \infty, \tag{7b}$$

$$\infty = J(0, 1) > J(m, 1) > J(m', 1) > J(1, 1) = 0. \tag{7c}$$

The monotonicity conditions mean that the informed trader earns higher expected profits when the asset is more mispriced. The boundary conditions mean that the informed trader earns zero future profit if his type is detected and infinite profits if the market believes his type to be the opposite of what it is.

Given the intensities with which the informed trader trades, it is easy to calculate the conditional expectations. For each order size i, define

$$\pi_i^+(m) = m\theta_i^+(m, 1) + (1 - m)\theta_i^+(m, 0) + \beta_i, \tag{8a}$$

$$\pi_i^-(m) = m\theta_i^-(m, 1) + (1 - m)\theta_i^-(m, 0) + \beta_i. \tag{8b}$$

These are the arrival intensities for buy and sell orders of size i, conditional on the liquidity providers' information. A simple Bayes's rule calculation (provided in the Appendix) yields the following:

PROPOSITION 1: *The expected value of the asset conditional on a buy order of size i at date t and given $m_{t-} = m$ is*

$$a_i(m) = \frac{m\theta_i^+(m,1) + m\beta_i}{\pi_i^+(m)}. \tag{8c}$$

Similarly, the expected value conditional on a sell order of size i is

$$b_i(m) = \frac{m\theta_i^-(m,1) + m\beta_i}{\pi_i^-(m)}. \tag{8d}$$

The expected value conditional on a buy order of size i or greater is

$$a_{i+}(m) = \frac{\sum_{j=i}^{n} \pi_j^+(m) a_j(m)}{\sum_{j=i}^{n} \pi_j^+(m)}. \tag{8e}$$

Likewise, the expected value conditional on a sell order of size i or greater is

$$b_{i+}(m) = \frac{\sum_{j=i}^{n} \pi_j^-(m) b_j(m)}{\sum_{j=i}^{n} \pi_j^-(m)}. \tag{8f}$$

Furthermore, the process (m_t) being a martingale relative to the liquidity providers' information implies

$$f(m) = m(1-m)\sum_{i=1}^{n}\left[\theta_i^+(m,0) + \theta_i^-(m,0) - \theta_i^+(m,1) - \theta_i^-(m,1)\right]. \tag{8g}$$

We conjecture that, perhaps under some auxiliary technical assumptions, conditions (6) to (8) are necessary for parts (i) and (ii) in the definition of equilibrium.[12] Theorem 2 of Back-Baruch (2004) generalizes easily to the present model and shows that (6) to (8) are sufficient conditions for (i) and (ii) to hold.[13]

[12] Conditions (6) and (8) are clearly necessary for a Markovian equilibrium (i.e., an equilibrium in which the the informed trader's intensities are functions of the conditional expectation m). We use the monotonicity conditions in (7b) and (7c), but we are unable to prove that they are necessary. The monotonicity means that the informed trader's expected profit is lower when the market becomes more nearly certain of his type. If there are equilibria that do not have this property (which we doubt), they are certainly pathological.

[13] We omit here a very mild technical condition in the informed trader's optimization problem. To ensure expected profits are well defined, a strategy is defined to be admissible in Back-Baruch (2004) if it does not incur infinite expected losses. That restriction on trading strategies should be imposed in the present model to establish the sufficiency result.

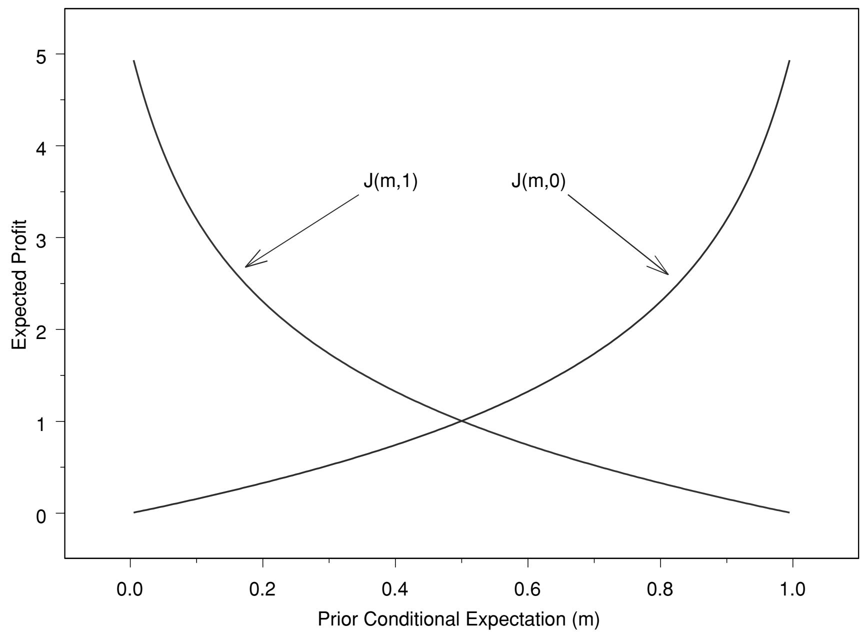

Figure 1. Value functions. This shows the value function $J(m, \tilde{v})$ of the informed trader in a limit order market for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$.

A consequence of (6) to (8) is that there must be more information content in larger orders.

PROPOSITION 2: *Assume conditions (6) to (8) hold for all* $1 \le i \le n$ *and all* $m \in (0, 1)$. *Then, for each* $j > i$ *and each* $m \in (0, 1)$, $a_j(m) > a_i(m)$ *and* $b_j(m) < b_i(m)$.

We attempted to solve conditions (6) to (8) numerically for $n = 2, 3, 4, 5$ and various parameter configurations, and we were successful in each case.[14] Figure 1 depicts the value functions $J(m, 0)$ and $J(m, 1)$ of the low- and high-type traders for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. Figure 2 presents the intensities $\theta_i^+(m, 1)$ of buy orders by the high-type informed trader for the same parameter values. Figure 2 shows that $\theta_1^+ < \theta_2^+ < \theta_3^+$ for the high type, which means that large orders are used more intensively than small orders, and thus there is greater information content in large orders, as shown in Proposition 2.

[14] The essence of the solution method is to iterate on conditions (6) and (7a). Given a guess for the value function, the equalities in condition (6) can be used to compute the ask and bid prices. Given the ask and bid prices, condition (7a) is a functional equation in the value function that can be used to update the guess. When this iteration has converged, the equilibrium order intensities of the informed trader can be computed from condition (8). We did not need to impose the monotonicity conditions in (7b) and (7c); in each case, they were automatically satisfied. Likewise, the inequality conditions in (6) were automatically satisfied with strict inequalities, that is, there was no "bluffing" as discussed in Back-Baruch (2004). For more details, see Appendix B of Back and Baruch (2004).

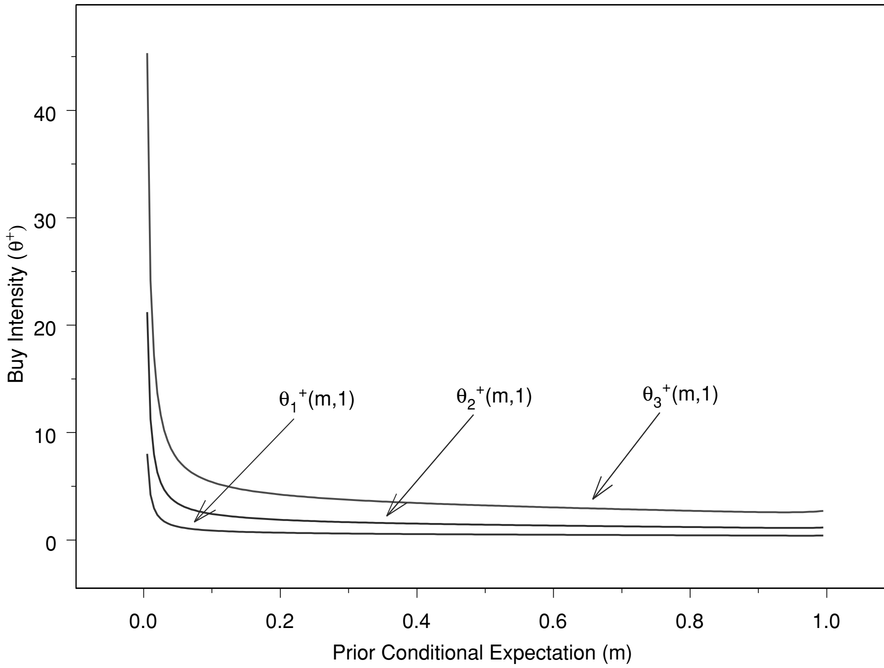

Figure 2. Intensities of trading. This shows the intensities of different buy order sizes for the informed trader in a limit order market when $\tilde{v} = 1$. The parameter values are $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. The ordering of the intensities is $\theta_1^+(m, 1) < \theta_2^+(m, 1) < \theta_3^+(m, 1)$, showing that larger order sizes are used more frequently (and implying that there is more information content in larger orders). The figure also illustrates that the intensity of buying increases when m decreases.

We discuss this numerical example further in Section V, where we show that it satisfies part (iii) of the definition of equilibrium. In fact, for $n = 2, 3, 4, 5$ and each parameter configuration we considered, the numerical solution of conditions (6) to (8) satisfies part (iii) of the definition of equilibrium. For $n = 2$ we can confirm analytically that this is true—see Section IV. Thus, it appears that there is always a block order equilibrium in a limit order market.

III. Uniform Price Markets

In this section, we present some basic facts about uniform price markets. As in the previous section, we assume here that liquidity traders submit block orders, ignoring part (iii) of the definition of equilibrium.

In a uniform price market, the profit earned by the informed trader on a buy order of size i at date t is

$$i\tilde{v} - ia_i^U(m_{t-}). \tag{9a}$$

Likewise, the profit earned on a sell order of size i is

$$ib_i^U(m_{t-}) - i\tilde{v}. \tag{9b}$$

The only difference in the equilibrium conditions in a uniform price market compared to the equilibrium conditions (6) to (8) in a limit order market is that the costs/revenues

$$\sum_{j=1}^{i} a_{j+}^L(m) \quad \text{and} \quad \sum_{j=1}^{i} b_{j+}^L(m) \tag{10a}$$

in condition (6) should be replaced by

$$ia_i^U(m) \quad \text{and} \quad ib_i^U(m), \tag{10b}$$

respectively, in a uniform price market. We repeat condition (6) here, making these substitutions and dropping the U superscript:

$$J(m, 1) = i - ia_i(m) + J(a_i(m), 1), \tag{11a}$$

$$J(m, 1) \geq ib_i(m) - i + J(b_i(m), 1), \text{with equality when } \theta_i^-(m, 1) > 0, \tag{11b}$$

$$J(m, 0) = ib_i(m) + J(b_i(m), 0), \tag{11c}$$

$$J(m, 0) \geq -ia_i(m) + J(a_i(m), 0), \text{with equality when } \theta_i^+(m, 0) > 0. \tag{11d}$$

The Bayes's rule calculations in Proposition 1 also apply to uniform price markets. Furthermore, uniform price markets share the characteristic of limit order markets that larger orders must have more information content than small orders.

PROPOSITION 3: *Assume conditions (7), (8), and (11) hold for all* $1 \leq i \leq n$ *and all* $m \in (0, 1)$. *Then, for each* $j > i$ *and each* $m \in (0, 1)$, $a_j(m) > a_i(m)$ *and* $b_j(m) < b_i(m)$.

We can solve the equilibrium conditions for the uniform price model, ignoring part (iii) of the definition of equilibrium, in the same way that we solve the limit order model. We present a solution for the case $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$ in Section V. However, in Section V, we also show that, in contrast to the limit order market, this numerical solution does not satisfy part (iii) of the definition of equilibrium. In fact, we show that any solution of conditions (7), (8), and (11) has the property that it must be cheaper for some liquidity traders to work their orders. We begin our analysis of this issue in the next section.

IV. Working Orders: Two Order Sizes

In this and the following section we focus on part (iii) of the definition of equilibrium and ask whether liquidity traders can reduce their execution costs by working orders. In a limit order market, submitting market orders with such little time between them that no new limit orders arrive and none are canceled will cause one to hit the successive limit prices, producing the same execution as a block order. However, we assume that liquidity providers monitor the book continuously, so that a market order trader need only wait an instant between orders for the book to be replenished. We will show that working orders in a limit order market, waiting for the book to be replenished between orders, *does not* provide better execution than block orders; however, working orders in a uniform price market *does* provide better execution.

The case $n = 2$ that we consider in this section is special because it does not admit the possibility of partial pooling. We present an example of a partially pooling equilibrium in a uniform price market when $n = 3$ in Section VI. In that example, size 3 traders pool with size 1 traders and size 2 traders separate, that is, size 3 traders work orders and size 2 traders submit blocks. In contrast, when $n = 2$ there are only two possibilities at any date t and for any $m = m_{t-}$: either size 2 traders submit blocks (separate) or work orders (pool).

Our approach is to assume initially that liquidity traders submit block orders and that the results of the previous two sections apply. We then compare the cost of submitting a block order to the cost of working orders.

At any date t, the arrival of a small buy order at date t produces an updating of expectations denoted by $m_t = a_1(m_{t-})$, which we abbreviate to a_1 in the table below. If a second small buy order is submitted a very short time afterwards, then with probability arbitrarily close to one there will be no intervening order and the expected value of the asset prior to receipt of the second buy will be arbitrarily close to a_1. Receipt of the second buy will cause beliefs to change to $a_1(a_1(m_{t-}))$, which we abbreviate to $a_1(a_1)$. Using similar notation throughout, we can compare the cost of a large buy to two small buys submitted very close together in the two market types (UPM = uniform price market and LOM = limit order market) as follows:

Market Type	Cost of Large Order	Cost of Two Small Orders	Difference in Costs
UPM	$2a_2^U$	$a_1^U + a_1^U(a_1^U)$	$2a_2^U - a_1^U - a_1^U(a_1^U)$
LOM	$a_{1+}^L + a_2^L$	$a_{1+}^L + a_{1+}^L(a_1^L)$	$a_2^L - a_{1+}^L(a_1^L)$

In computing the costs for the limit order market, we use the fact that $a_{2+}^L = a_2^L$, because 2 is the maximum order size. The right-hand side of the table shows the difference in the costs of a large buy versus two small buys in the two markets. Because the informed trader cares about both execution costs and the effects of trades on the market's beliefs (which determine his expected future trading profits), the critical comparison turns out to be between (i) the difference in the costs of a large buy versus two small buys and (ii) the difference in the updated expectations resulting from a large buy versus two small buys. This comparison is as follows:

Market Type	Difference in Costs Compared to Difference in Expected Values
UPM	$2a_2^U - a_1^U - a_1^U(a_1^U) > a_2^U - a_1^U(a_1^U)$
LOM	$a_2^L - a_{1+}^L(a_1^L) < a_2^L - a_1^L(a_1^L)$

To deduce the inequalities we use only the facts that $a_2^U > a_1^U$ and $a_{1+}^L > a_1^L$, which follow from Propositions 2 and 3 (large orders have more information content than small orders). The different relations ($>$ for a UPM and $<$ for a LPM) between the difference in costs and the difference in expectations are responsible for the differences in the two markets regarding the optimal behavior of discretionary liquidity traders, as we will see. A large buy is significantly more expensive than two small buys in a uniform price market (the difference in costs is greater than the difference in expected values), so only small orders should be used in a uniform price market; however, the opposite is true in a limit order market.

We repeat here the first-order condition (11a) for the high-type trader in a uniform price market for $i = 1, 2$. To reduce the notational burden, we omit the U superscript on the a_i. Note that the value functions also depend on the market type; we suppress this notation as well.

$$J(m, 1) = 1 - a_1 + J(a_1, 1), \tag{12a}$$

$$J(m, 1) = 2 - 2a_2 + J(a_2, 1). \tag{12b}$$

The first equality holds for each m; therefore, it also holds at a_1. Substituting this fact for $J(a_1, 1)$ in the first line gives us

$$J(m, 1) = 2 - a_1 - a_1(a_1) + J(a_1(a_1), 1). \tag{12c}$$

Subtracting equation (12c) from equation (12b) yields

$$J(a_2, 1) - J(a_1(a_1), 1) = 2a_2 - a_1 - a_1(a_1). \tag{12d}$$

Equation (12d) simply says that the difference in the continuing values, from one large buy relative to two small buys, must equal the difference in the costs. Given the inequality in the second table above we have

$$J(a_2, 1) - J(a_1(a_1), 1) = 2a_2 - a_1 - a_1(a_1) > a_2 - a_1(a_1). \tag{12e}$$

The value function $J(\cdot, 1)$ must be a decreasing function (high prices are bad for the trader with good news) so the left and right-hand sides of (12e) must have opposite signs. Therefore,

$$2a_2 - a_1 - a_1(a_1) > 0 > a_2 - a_1(a_1). \tag{13}$$

The first inequality in (13) states that two small buys are cheaper than one large buy in a uniform price market.

Analogous reasoning for the limit order market yields the opposite result, because of the opposite inequality in the second table above. As in a uniform price market, the difference in continuing values for the high-type trader must equal the difference in the costs (this is a consequence of the optimality condition (6a)). Thus,

$$J(a_2, 1) - J(a_1(a_1), 1) = a_2 - a_{1+}(a_1) < a_2 - a_1(a_1). \tag{14}$$

Because $J(\cdot, 1)$ is a decreasing function, the left- and right-hand sides of (14) must have opposite signs. Therefore,

$$a_2 - a_{1+}(a_1) < 0 < a_2 - a_1(a_1). \tag{15}$$

The first inequality in (15) shows that the cost of one large buy $(a_{1+} + a_2)$ is less than the cost of two small buys $(a_{1+} + a_{1+}(a_1))$ in a limit order market.

To further clarify the origin of the results, it may be useful to note that in both markets we have the following inequalities:

$$a_{1+}(a_1) > a_2 > \frac{a_1 + a_1(a_1)}{2}. \tag{16}$$

The first inequality in (16) states that a block order is cheaper in a limit order market; the second inequality states that working orders is cheaper in a uniform price market. These facts we have already discussed. However, the first inequality also holds in a uniform price market—by virtue of the second inequality in (13) and the fact that $a_{1+}(a_1) > a_1(a_1)$—and the second inequality also holds in a limit order market—by virtue of the second inequality in (15) and the fact that $a_2 > a_1$. Thus, to some extent, it is not the differences in the conditional expectations in the two markets that drive the different results. Rather, it is simply the way that execution prices are determined. Prices in uniform price markets are conditional expectations (rather than conditional tail expectations). Thus, as we noted before, prices in uniform price markets "should" be better for small orders than are prices in limit order markets. Consequently, it is cheaper to submit two small buys than one large buy in a uniform price market, and the reverse is true in a limit order market. To put this another way, the "unfair" prices for small orders in a limit order market cause large orders to be incentive compatible, whereas they are not incentive compatible in a uniform price market.

V. Working Orders: The General Case

We continue to examine part (iii) in the definition of equilibrium, but now for general n. From the previous section, we know for $n = 2$ that there is a block order equilibrium in a limit order market[15] but no block order equilibrium in a uniform price market. The result for uniform price markets extends to general n as follows: There is never an equilibrium in which all market order traders submit block orders.

For limit order markets, we are only able to obtain numerical results. To show that liquidity traders obtain better execution with blocks than by working orders, we need to show for each order size i and each series of order sizes $i_1, \ldots, i_k$ such that $i_1 + \cdots + i_k = i$ that the cost

$$\sum_{j=1}^{i} a_{j+}(m) \tag{17a}$$

of a buy order of size i is less than the cost

$$\sum_{j=1}^{i_1} a_{j+}(m) + \sum_{j=1}^{i_2} a_{j+}(a_{i_1}(m)) + \cdots + \sum_{j=1}^{i_k} a_{j+}(a_{i_{k-1}}(a_{i_{k-2}}(\cdots a_{i_1}(m)))) \tag{17b}$$

of submitting sequential buy orders of sizes $i_1, \ldots, i_k$. As mentioned before, the numerical results indicate that the block order is always cheaper. Figure 3 presents the case of $n = 3$ with the same parameter values as the previous figures ($r = \beta_1 = \beta_2 = \beta_3 = 1$).

Consider, for example, $m = 0.2$ (shown in Figure 3 and Table I) and consider the cost of an order of size 3 relative to an order of size 2 and then an order of size 1. The cost of the third unit in the block order of size 3 is $a_3(0.2)$, which is 0.5668. The cost of the third unit when the order is split is $a_{1+}(a_2(0.2))$. The conditional expectation following the order of size 2 is $a_2(0.2) = 0.4191$. The inside ask quote following the order of size 2 is $a_{1+}(0.4191)$, which is $0.6710 > 0.5668$. This example generalizes: We have compared the costs (17a) and (17b) and verified numerically that the cost of the block order is smaller, for as many as five order sizes, for various values of the parameter vector $(r, \beta_1, \ldots, \beta_n)$, for each possible split of each order size, and for each value of m. Thus, we conclude that in limit order markets better execution is obtained with block orders. This (apparent) fact is unsurprising: There is no benefit in breaking a large order into pieces and executing the pieces against limit prices for small orders, because those prices already anticipate execution against larger market orders.

The situation in uniform price markets is very different. We can establish analytically that at least some orders must be worked in uniform price markets. The hypothesis of the following theorem must hold in any equilibrium of a uniform price market in which all orders are block orders. The right-hand side of (18a) is the (approximate) cost of working a buy order of size i by first submitting an order of size j and then submitting an order of size $i - j$. The inequality shows

[15] More precisely, we know that part (iii) of the definition of equilibrium follows automatically when conditions (6) to (8) for a block order equilibrium are satisfied.

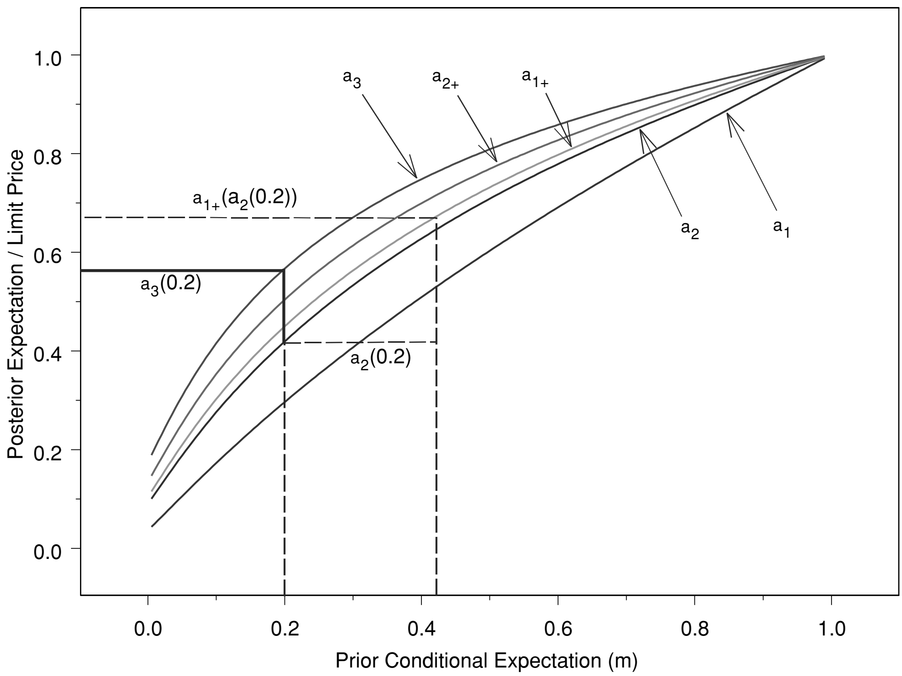

Figure 3. Expectations and ask prices. This shows the conditional expectations a_1, a_2, a_{1+}, a_{2+}, and $a_3 = a_{3+}$ in a limit order market for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. The ordering is $a_1 < a_2 < a_{1+} < a_{2+} < a_3$. The highlighted points illustrate the difference between the pricing of a block order of size 3 and the pricing of a buy order of size 2 followed by a buy of size 1, when the initial conditional expectation is $m = 0.2$. The third unit in a block buy of size 3 is priced at $a_3(0.2)$, whereas a buy of size 1 following a buy of size 2 is priced at $a_{1+}(a_2(0.2)) > a_3(0.2)$.

that working orders is cheaper. Thus, the theorem shows, by contradiction, that there are no equilibria with exclusively block orders in a uniform price market.

THEOREM 1: *Assume conditions (7), (8), and (11) hold for all* $1 \leq i \leq n$ *and all* $m \in (0, 1)$. *Then, for each* $i > 1$, *each* $j < i$, *and each* $m \in (0, 1)$, *we have*

$$ia_i(m) > ja_j(m) + (i-j)a_{i-j}(a_j(m)). \tag{18a}$$

$$ib_i(m) < jb_j(m) + (i-j)b_{i-j}(b_j(m)). \tag{18b}$$

The conclusion of the theorem is that it is cheaper to work any market order by splitting it into pieces. However, the theorem does not imply that all market orders must be worked in an equilibrium of a uniform price market, because the hypothesis of the theorem is that the market is in a block order equilibrium. Once we admit the possibility that some orders may be worked and some may be submitted as blocks, violating assumption (11), the conclusion of the theorem

Table I

Snapshots of the Limit Order Book

This shows the limit order book for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. The left column is the book when $m = 0.2$. Following a buy order of size 2, the conditional expectation changes to $m = a_2(0.2) = 0.4191$. The right column is the new book at $m = 0.4191$.

Initial Book	Book Following a Buy of Size 2
$a_3 = 0.5668$	$a_3 = 0.7610$
$a_{2+} = 0.5037$	$a_{2+} = 0.7142$
$a_{1+} = 0.4497$	$a_{1+} = 0.6710$
$m = 0.2$	$m = 0.4191$
$b_{1+} = 0.0923$	$b_{1+} = 0.2150$
$b_{2+} = 0.0777$	$b_{2+} = 0.1837$
$b_3 = 0.0629$	$b_3 = 0.1510$

need not hold. In the next section, we will give an example of an equilibrium in a uniform price market in which some, but not all, orders are worked.

To illustrate the theorem, we can solve conditions (7), (8), and (11) and then compare the cost of block orders to the cost of worked orders. Figure 4 and Table II pertain to the case $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. Consider for example $m = 0.2$. Figure 3 and Table I show that it is cheaper in a limit order market (for the same parameter values) to submit a block buy of size 3 than to first submit a buy of size 2 and then a buy of size 1. Figure 4 and Table II show that the opposite is true in a uniform price market. The cost of a buy of size 2 when $m = 0.2$ is shown in Table II as $2 \times 0.4442 = 0.8884$. Following a buy of size 2, m changes to 0.4442 and the cost of a buy of size 1 is 0.5923. The total cost of this strategy is therefore $0.8884 + 0.5923 = 1.4807$. In contrast, the cost of a block buy of size 3 when $m = 0.2$ is $3 \times 0.5386 = 1.6158 > 1.4807$. In fact, submitting a buy of size 2 and then a buy of size 1 is not the cheapest way to execute a buy demand of size 3—it is even better to work the order as three successive single-unit orders. With these parameter values, the cost of three successive single-unit buy orders when $m = 0.2$ is $0.3258 + a_1(0.3258) + a_1(a_1(0.3258) = 0.3258 + 0.4732 + 0.6191 = 1.4181$. Therefore, the numerical solution shown in Figure 4 and Table II does not satisfy part (iii) of the definition of an equilibrium: Submitting block orders is suboptimal behavior for liquidity traders.

VI. A Positive Theory of Working Orders

To this point, we have only shown for uniform price markets that there are no equilibria in which all orders are block orders; thus, some orders must be worked in equilibrium if an equilibrium exists. However, we have not presented a model of working orders. To do so, and to show that there are equilibria in which orders are worked, is the purpose of this section.

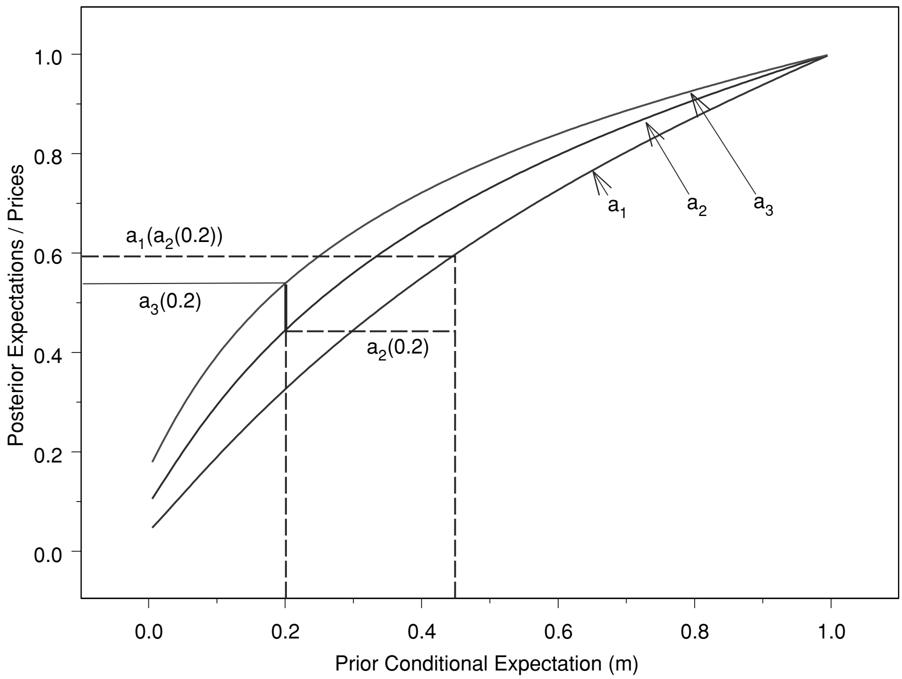

Figure 4. Expectations and uniform prices. This shows the conditional expectations a_1, a_2, and a_3 in a uniform price market (assuming block orders) for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. The ordering is $a_1 < a_2 < a_3$. The highlighted points illustrate the difference between the pricing of a block buy order of size 3 and the pricing of a buy order of size 2 followed by buy of size 1, when the initial conditional expectation is $m = 0.2$. A buy order of size 3 is priced at $3a_3(0.2) = 1.6158$, whereas the cost of a buy of size 2 followed by a buy of size 1 is $2a_2(0.2) + a_1(a_2(0.2)) = 1.4807$.

Because we do not model the preferences and constraints of liquidity traders, we are unable to determine the optimal timing of their successive orders if they choose to work their orders. However, because we are working in continuous time and hence there is no fixed amount of time that must elapse between submission of two successive orders, it is possible to assume that liquidity traders obtain execution at the instant they arrive at the market even though they submit a series of orders. This modeling choice allows liquidity traders discretion over the method of execution while retaining the assumption that they have inelastic demands at an instant in time.

We intend for the model to be understood as the limit of a model in which orders are submitted very close together. The sequencing of orders can matter, just as if it would if the orders were submitted at different times. For example, the sequence (1, 2), meaning a single-unit order and then an order for two units, may result in different pricing and beliefs than the sequence (2, 1). Interpreting the model as a limit restricts beliefs conditional on certain out-of-equilibrium events. This can most easily be explained via an example. Suppose that in

Table II

Snapshots of Uniform Prices

This shows posterior conditional expectations (uniform prices) for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$, assuming block orders. The left column shows the posterior expectations when $m = 0.2$. Following a buy order of size 2, the conditional expectation changes to $m = a_2(0.2) = 0.4442$. The right column shows the posterior expectations at $m = 0.4442$.

Initial Prices	Prices Following a Buy of Size 2
$a_3 = 0.5386$	$a_3 = 0.7513$
$a_2 = 0.4442$	$a_2 = 0.6889$
$a_1 = 0.3258$	$a_1 = 0.5923$
$m = 0.2$	$m = 0.4442$
$b_1 = 0.1270$	$b_1 = 0.3090$
$b_2 = 0.0922$	$b_2 = 0.2311$
$b_3 = 0.0722$	$b_3 = 0.1831$

equilibrium every trader is supposed to submit blocks but a pair of single-unit buy orders arrive at the same time. This is an out-of equilibrium event in our limit model, but it is consistent with equilibrium before reaching the limit, because it could represent orders from two different traders (or two orders from the informed trader submitted close together). Therefore, we require liquidity providers to update beliefs as if they received two distinct single-unit orders, that is, as $a_1(a_1(m))$. Note that there is no restriction on beliefs in the reverse case: If orders are supposed to be worked in equilibrium, our equilibrium concept does not restrict beliefs conditional on observing a block.

Our first goal is to explain why there is always a pooling equilibrium in a uniform price market that is equivalent to a block order equilibrium in a limit order market. By "pooling" we mean that all traders work their orders as a series of single-unit orders. It is convenient to modify our notation somewhat. Let $a^U_{j+}(m)$ and $b^U_{j+}(m)$ denote the conditional expectations at date t given $m_{t-} = m$ and conditional on there being j or more single-unit buy or sell orders, respectively, at date t (i.e., conditional on observing j orders without knowing whether more orders are coming at date t). The cost/revenue from submitting i single-unit buy/sell orders at date t is then

$$\sum_{j=1}^{i} a^U_{j+}(m) \quad \text{and} \quad \sum_{j=1}^{i} b^U_{j+}(m). \tag{19}$$

The conditional expectations a^U_{j+} and b^U_{j+} obviously correspond to limit prices in a limit order market, because they are conditioned on the complete demand at date t being of size j or larger. The equivalence between block order equilibria in limit order markets and pooling (worked order) equilibria in uniform price markets follows almost immediately. In the following, $a^U_i(m)$ and $b^U_i(m)$ denote the conditional expectations given exactly i single-unit buy or sell orders, respectively, at any date t with $m = m_{t-}$.

THEOREM 2: *Let $a_i^L(m)$ and $b_i^L(m)$ be equilibrium conditional expectations in a limit order market. Then there is an equilibrium in a uniform price market in which $a_i^U(m) = a_i^L(m)$ and $b_i^U(m) = b_i^L(m)$ for all m and in which all demands by liquidity traders and the informed trader are worked as a series of single-unit orders, with the timing of demands being the same as in the limit order market. Conversely, if $a_i^U(m)$ and $b_i^U(m)$ are equilibrium conditional expectations in a uniform price market in which all demands by liquidity traders and the informed trader are worked as a series of single-unit orders, then $a_{i+}^L(m) = a_{i+}^U(m)$ and $b_{i+}^L(m) = b_{i+}^U(m)$ are equilibrium limit prices in a limit order market, and the equilibrium strategy of the informed trader is the same in the limit order market as in the uniform price market (with the exception that block trades in the limit order market are executed as a series of single-unit orders in the uniform price market).*

Proof: To establish the first part, we need to show that neither the liquidity traders nor the informed trader wish to deviate. The informed trader has more choices in this uniform price market than in the limit order market: He can work orders as a series of single-unit orders, in which case execution is identical to the limit order market, or he can submit block orders. To ensure he does not wish to deviate by submitting block orders, we need to define beliefs conditional on block orders (which are out-of-equilibrium events). A particular set of beliefs that will support the equilibrium is that the likelihood of a block of size i having been submitted by an informed trader is the same as the likelihood of a worked order of size i or larger coming from an informed trader and moreover that the block will be followed by orders in the same direction from the same trader with the same likelihood as if the block had been i single unit orders. This implies that a block buy of size i will be executed at $a_{i+}^U(m)$, and, if it is not immediately followed by more orders in the same direction, beliefs will settle at $a_i^U(m)$. With these beliefs, continuation values are the same from block orders as from a series of single-unit orders, and single-unit orders have lower costs (higher revenues); this is exactly the same as saying that walking up the limit order book is better than executing at the marginal limit price. Therefore, the informed trader will not use block orders. The informed trader's opportunities in submitting i single-unit orders at any point in time, for various i, are exactly the same as the opportunities in submitting block orders of size i in the limit order market. Therefore, the equilibrium behavior in the limit order market is also equilibrium behavior (with orders worked as a series of single-unit orders) in the uniform price market.

Liquidity traders can deviate by submitting block orders or by delaying slightly the submission of single-unit orders. With beliefs, and hence prices, defined as in the previous paragraph, block orders are suboptimal for liquidity traders as well as for the informed trader. The concept of delaying orders is as follows. Consider submitting, for example, a series of two buy orders with the second slightly delayed. Liquidity providers will not interpret the second as being part of the same demand, so they will update expectations as $a^U_{1+}(a^U_1(m))$ rather than as $a^U_{2+}(m)$. Liquidity traders have the same opportunity in the limit order market. If they prefer to submit a block order in the limit order market, then it must be that $a^L_{1+}(a^L_1(m)) \geq a^L_{2+}(m)$, which, since we have defined $a^U = a^L$, implies that delaying slightly the submission of single-unit orders is not a superior strategy in the uniform price market. The general case is the same: Liquidity traders have the same opportunities in the two markets, so if they obtain the best execution with block orders in the limit order market, they obtain the best execution in the uniform price market by submitting a series of single-unit orders that are recognized by liquidity providers as being part of the same demand.

The proof of the converse is even simpler, because, with all order sizes being used with positive probability in the limit order market, we do not need to define out-of-equilibrium beliefs. Liquidity providers do not anticipate orders being worked as a series of single-unit orders, given the behavior prescribed for the informed trader and liquidity traders, so they update beliefs given a series of single-unit orders as $a^i_1(m)$ or $b^i_1(m)$, where i is the number of single-unit orders and $a^i(b^i)$ denotes iteration of the map $a(b)$. The opportunities for the informed trader and the liquidity trader are the same in the limit order market as in the uniform price market, so the equilibrium in the uniform price market is an equilibrium in the limit order market. Q.E.D.

Partially pooling equilibria can arise in a uniform price market because, while mid-size traders will want to pool with small traders, they will not want to pool with large traders; thus, they may separate in equilibrium. We give an example of this for the parameters $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. Figures 1–3 illustrate a block order equilibrium in a limit order market for these parameter values. By Theorem 2, this is also a worked order equilibrium in a uniform price market. However, there are two equilibria in this uniform price market, the other equilibrium being partially pooling.

The equilibrium prices in the partially pooling equilibrium are shown in Figure 5. In this equilibrium, single-unit orders are submitted only by traders desiring to trade a single unit and by traders desiring to trade three units. Size 2 traders pay $2a_2(m)$ when submitting a block buy order. Size 3 traders submit first a single-unit order and then a block of size 2. A single-unit buy order receives the partially pooling price $a_{1+}(m)$, which is the conditional expectation given that the demand underlying the order is of size 1 or size 3. The block of size 2 identifies the trader as having a demand of 3 units, and the total cost of the three units is $a_{1+}(m) + 2a_3(m)$.

A size 2 buyer who deviates to working orders pays $a_{1+}(m)$ for the first unit. Upon receiving an order for a second unit, market makers infer that the order for the first unit was submitted by a buyer with a single-unit demand and that the order for the second unit is from another buyer, whose total demand may be either one or three units. Thus, the total cost of the two units is $a_{1+}(m) + a_{1+}(a_1(m))$. It turns out that the block order is cheaper, for all values of m. In

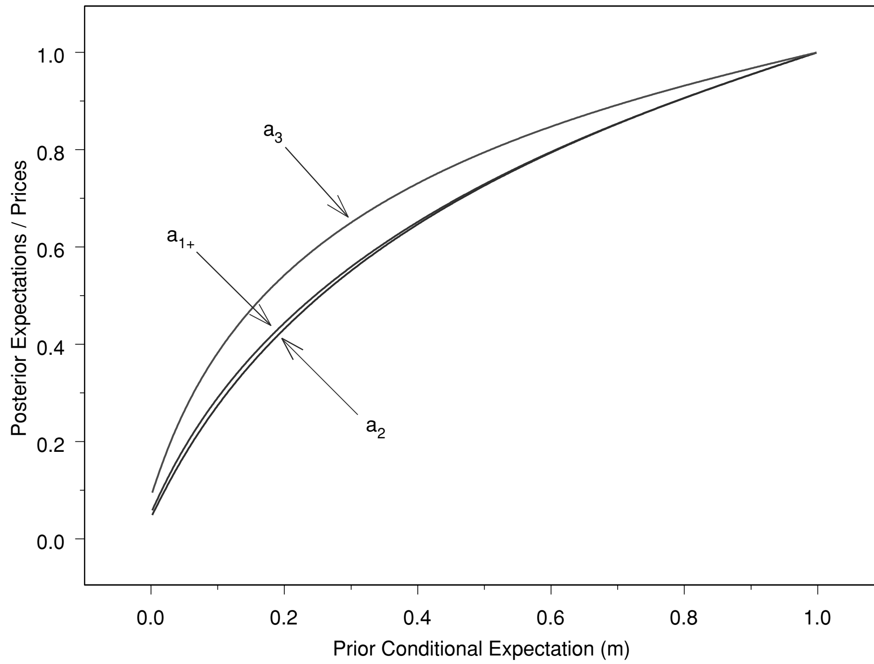

Figure 5. Partially pooling equilibrium. This shows the partially pooling equilibrium in a uniform price market for the parameter values $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$. A single-unit order executes at $a_{1+}(m)$ and an order for two units executes at $a_2(m)$. For these parameters, $a_2 < a_{1+}$. The size 3 trader works his buy order at cost $a_{1+} + 2a_3$.

fact, as Figure 5 shows, $a_{1+}(m) > a_2(m)$,[16] so, because $a_{1+}(a_1(m)) > a_{1+}(m)$, it must be the case that $a_{1+}(m) + a_{1+}(a_1(m)) > 2a_2(m)$.

A size 3 trader could deviate by submitting a block order. Blocks of size 3 are out-of-equilibrium events, but we can enforce the equilibrium by defining the expected value conditional on a block as being $a_3(m)$. Working orders is cheaper than submitting blocks, because $a_{1+}(m) + 2a_3(m) < 3a_3(m)$. The size 3 trader could alternatively work his order as first a size 2 order and then a size 1 order, at cost $2a_2(m) + a_{1+}(a_2(m))$, or as a sequence of size 1 orders, at cost $a_{1+}(m) + a_{1+}(a_1(m)) + a_{1+}(a_{1+}(a_1(m)))$, but these strategies and all other possible deviations are also more expensive. Thus, the strategies described form an equilibrium.

Partially pooling equilibria in a uniform price market are less robust than the fully pooling equilibrium in the sense that liquidity traders must know the exact number of possible order sizes and the other parameter values in order to know whether to submit blocks or to work orders. For example, the

[16] This means that small orders have more of a price impact than mid-size orders. This is perverse but theoretically possible on a floor exchange. Reiss and Werner (1996) show that large trades get better prices than small trades on the London Stock Exchange. Partial pooling is a possible theoretical explanation, but one that seems less plausible than that given by Bernhardt, Dvoracek, Hughson, and Werner (2005), since trading on the London Stock Exchange is not anonymous.

partially pooling equilibrium described in the previous paragraph for $n = 3$ and $r = \beta_1 = \beta_2 = \beta_3 = 1$ disappears if $n = 4$: We confirmed numerically that there is no equilibrium for these parameter values (and $\beta_4 = 1$) when $n = 4$ in which size 2 traders separate and size 3 traders pool with size 1 traders. In contrast, the fully pooling equilibrium involves all traders working orders as single-unit orders, regardless of the number of possible order sizes or other parameter values.

VII. Hybrid Markets

Rock (1990) models a hybrid market consisting of a floor exchange and a limit order book. In Rock's model, floor traders can provide price improvement for market orders by stepping ahead of the book by one tick (which in his model is infinitesimal) after observing the size of a market order. As is now very well understood, this "penny jumping" imposes adverse selection on limit order traders, reducing the supply of liquidity in the limit order book.[17]

Rock assumes the floor consists of a risk-averse monopolist specialist. If we assume the specialist is actually risk neutral, then the adverse selection imposed on limit order traders is more severe. Consider our model of a limit order market but including a risk-neutral specialist. There is an equilibrium in which market order traders all submit block orders, the execution price of each buy order is (epsilon less than) a_n, and the execution price of each sell order is (epsilon more than) b_n, where, as before, a_n and b_n denote the extreme conditional expectations, that is, the expectations conditional on the largest possible buy and sell orders, respectively. In this equilibrium, it is simply impossible for limit order traders to compete with the specialist, so the specialist makes large profits. To see this, consider submitting a limit sell order at a price $a < a_n$. Given any market order such that the expected asset value is $a' < a$, the specialist will price improve on the limit order to take the market order at $a - \varepsilon$ for an arbitrarily small ε. Thus, the limit sell will only execute if the expected asset value is $a' \geq a$, implying that it will lose money.

Given that every buy order executes at a_n, there is no benefit to an individual trader to deviate from this block-order equilibrium and work his order. However, there is another equilibrium in which every market order trader works his order and execution prices are much better. In fact, there is a worked order equilibrium that is equivalent to a block order equilibrium in a limit order market without a floor. In other words, Theorem 2 also applies to a hybrid market consisting of a limit order book and a monopolist specialist (regardless of whether the specialist is risk-averse or risk-neutral). The reason is that, when all orders are worked, the specialist has no information advantage: In a worked order equilibrium, neither the specialist nor limit order traders can condition

[17] Empirical evidence for this phenomenon is provided by the move to decimalization. Bessembinder (2003) shows that the frequency of price improvement on the NYSE increased after decimalization, and Chakravarty, Wood, and Van Ness (2004) show that the depth in the limit order book on the NYSE declined. Goldstein and Kavajecz (2000) show that this also occurred when the NYSE moved from quoting in eighths to quoting in sixteenths.

on the size of the *demand* underlying a market order, and only one order size is ever observed (the smallest size). Therefore, information is symmetric between the specialist and limit order traders, and limit order traders can compete effectively. Except for the inside quotes, the limit order book is not uniquely defined in this equilibrium, but we can take it to be the same, for example, as in a block order equilibrium of a pure limit order market. Given this specification, market order traders will be indifferent between submitting block orders or working orders. In equilibrium, they work orders, simply walking up the book when they do so.

An alternative to assuming a monopolist specialist is to assume the floor consists of competitive floor traders (perhaps in conjunction with a specialist). This is the assumption made by Seppi (1997). If the floor traders were risk neutral and there were a block order equilibrium, then the limit order book would be essentially empty (consisting only of offers at a_n and bids at b_n) and each market order would execute at the expected value conditional on the order size. The limit order book would be empty for the same reason that it is empty in a block order equilibrium when there is a monopolist specialist: Floor traders will price improve whenever a limit order is profitable, implying that every limit order, except for those at extreme prices, must lose money. Execution prices would be conditional expected values in this model because of competition between floor traders. In other words, a hybrid market consisting of a limit order book and a competitive floor should be equivalent to a floor exchange. However, we already know (Theorem 1) that there is no block order equilibrium in this environment. Every equilibrium must involve at least partial pooling. This affects the equilibrium limit order book.

Theorem 2 also applies to this market: Regardless of whether the floor consists of a monopolist specialist or competitive floor traders, there is a worked order equilibrium in a hybrid market. This equilibrium is equivalent to a block order equilibrium of a pure limit order market.

VIII. Conclusion

Assuming perfect competition in liquidity provision, a single informed trader, and an infinitesimal tick size, we obtain two main results: (i) Every equilibrium in a uniform price market must involve at least partial pooling of large market order traders with small market order traders, with pooling occurring via the working of orders by large traders, and (ii) there is a fully pooling (worked order) equilibrium in a uniform price market that is equivalent to a block order equilibrium in a limit order market. The second result depends on modeling the working of orders as the submission of one order immediately after another, with no time elapsing between successive orders. However, the first result does not depend on this modeling device: Even if a nonzero (but arbitrarily small) amount of time must elapse between successive orders, there is no equilibrium in a uniform price market in which all orders are block orders. This shows that previous research on market design, which takes order sizes to be exogenous and constant across market types, is incomplete in an important way.

The assumption that orders can be worked with no time elapsing between successive orders probably merits some further discussion. The model should be understood as the limit of a model in which orders are worked very quickly. In such a model, given that orders from other traders arrive according to intensities, two orders appearing sufficiently close together will be viewed as coming from the same trader with high probability. Thus, the second order will be priced approximately as a second order from the same trader rather than as an order from a different trader. We only solve this model in the limit, in which market makers can infer exactly when two successive orders come from the same trader. It is important to note that this is not the same as allowing liquidity providers to infer the identity of a trader—in that case, reputational considerations would come into play. Liquidity providers in our model can determine when two trades come from the same trader, but the trades are anonymous and liquidity providers do not know the identity of the trader (in particular, they do not know, of course, whether the trader is informed or uninformed).

It may seem somewhat counter-intuitive that traders would choose to submit successive orders very close together. In some models, rapid execution leads to poor prices. For example, Brunnermeier and Pedersen (2005) argue that it takes time to bring liquidity to the market and hence rapid execution leads to greater price impacts. Our model is different in that we assume perfect competition in liquidity provision. Thus, we abstract from any illiquidity issues other than those arising from adverse selection.

When working orders, a large trader has some discretion in our model about the timing of orders: He may submit the orders close together and have them recognized as coming from a single trader, or he may delay and fool the market into thinking they were submitted by different traders. The informed trader in our model is indifferent about this choice. In equilibrium, he necessarily plays a mixed strategy, implying that it is optimal for him to trade at any time. In the worked order equilibrium, liquidity traders prefer that their orders be identified as coming from a single trader. This is the perhaps counter-intuitive aspect of our results, but it is less surprising when one considers that liquidity traders have the same choices in a limit order market: They can submit blocks, which are orders recognized as coming from a single trader, or they can fool the market by working orders. If they prefer to submit blocks in a limit order market, then they must prefer to work their orders very quickly in a uniform price market.

A key assumption of our model, as in Glosten (1994), is perfect competition in liquidity provision. Empirical studies such as Sandås (2001) and Biais, Bisiere, and Spatt (2003) show that competition by liquidity providers in limit order markets is not perfect in the sense we assume. These empirical results are consistent with the theoretical work of Bernhardt and Hughson (1997) and Biais, Martimort, and Rochet (2000), who show that perfect competition in a limit order market would require an infinite number of liquidity providers. Like other irrelevance propositions, our result on the equivalence of floor exchanges and limit order markets should be viewed as highlighting the issues that matter. In addition to strategic behavior by liquidity providers, we would look to risk aversion, reputational considerations, and the potential for front running as issues that may lead to different performance of limit order markets and floor exchanges. Our model should provide a benchmark for analyzing those issues.

Appendix: Proofs

Proof of Proposition 1: The probability of a buy order of size i arriving in an instant dt is

$$\pi_i^+(m)\,dt = m\theta_i^+(m,1)\,dt + (1-m)\theta_i^+(m,0)\,dt + \beta_i\,dt, \tag{A1}$$

where the three terms refer to the three possible sources of an order: the high-type informed trader, the low-type informed trader, and uninformed traders. The conditional expectation is the sum of the value-weighted conditional probabilities of the order coming from each of the three possible sources, namely,

$$\frac{m\theta_i^+(m,1)}{\pi_i^+(m)} \times 1 + \frac{(1-m)\theta_i^-(m,0)}{\pi_i^+(m)} \times 0 + \frac{\beta_i}{\pi_i^+(m)} \times m. \tag{A2}$$

This simplifies to equation (8c). The calculation for equation (8d) is similar, and these imply equations (8e) and (8f).

Considering only the jumps to a_i or b_i, the expected change in m in an instant dt given the information in the order flow and given $m_{t-} = m$ is

$$\begin{aligned}\sum_{i=1}^{n}(a_i(m)-m)\pi_i^+(m)\,dt + \sum_{i=1}^{N}(b_i(m)-m)\pi_i^-(m)\,dt \\ = m(1-m)\sum_{i=1}^{N}\left[\theta_i^+(m,1)+\theta_i^-(m,1)-\theta_i^+(m,0)-\theta_i^-(m,0)\right]dt.\end{aligned} \tag{A3}$$

This equation is a consequence of formulas (8c) and (8d). The process m is a conditional expectation, hence a martingale, so this expected change must be canceled by the expected change in m between transactions. This implies equation (8g). Q.E.D.

Proof of Proposition 2: We will give the proof for the buy side. It suffices to show that $a_i(m) > a_{i-1}(m)$ for each i. Note that (8a), (8c), and (8e) imply $a_{i+}(m) < 1$ for each $m \in (0,1)$. From (6a) we therefore have

$$J(a_i(m),1) - J(a_{i-1}(m),1) = a_{i+}(m) - 1 < 0. \tag{A4}$$

Assumption (7b) states that $J(\cdot,1)$ is a nonincreasing function, so the inequality (A4) implies $a_i(m) > a_{i-1}(m)$. Q.E.D.

Proof of Proposition 3: We will give the proof for the buy side. It suffices to show that $a_i(m) > a_{i-1}(m)$ for each i. Note that (8a) and (8c) imply $a_i(m) < 1$ for each $m \in (0, 1)$. From (11a) we therefore have

$$J(a_i(m), 1) - J(a_{i-1}(m), 1) = ia_i(m) - (i-1)a_{i-1}(m) - 1 \tag{A5a}$$

$$< i[a_i(m) - a_{i-1}(m)]. \tag{A5b}$$

Assumption (7b) states that $J(\cdot, 1)$ is a nonincreasing function, so the right-hand side of (A5) must be positive, that is, $a_i(m) > a_{i-1}(m)$. Q.E.D.

Proof of Theorem 1: We will give the proof for the buy side. From (11a) we have

$$J(a_i(m), 1) = J(m, 1) - i + ia_i(m), \tag{A6a}$$

and applying (11a) twice yields

$$J(a_{i-j}(a_j(m)), 1) = J(m, 1) - i + ja_j(m) + (i-j)a_{i-j}(a_j(m)). \tag{A6b}$$

Subtracting (A6b) from (A6a) and using the fact that $a_j(m) < a_i(m)$ yields

$$J(a_i(m), 1) - J(a_{i-j}(a_j(m)), 1) = ia_i(m) - ja_j(m) - (i-j)a_{i-j}(a_j(m)) \tag{A7a}$$

$$> (i-j)[a_i(m) - a_{i-j}(a_i(m))]. \tag{A7b}$$

Assumption (7b) states that $J(\cdot, 1)$ is a decreasing function, so the left- and right-hand sides of (A7) must have opposite signs. The left-hand side being positive implies

$$ia_i(m) > ja_j(m) + (i-j)a_{i-j}(a_j(m)). \tag{A8}$$

Q.E.D.

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