Tail Expectation, Imperfect Competition, and the Phenomenon of Flickering Quotes in Limit Order Book Markets

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Abstract

Perfect competition in liquidity provision in limit order markets is characterized by a tail expectation condition (Glosten 1994). In this paper, we model imperfect competition in schedules by infinitely many liquidity suppliers, quoting on a limit order book. We show that there are zero-rent mixed-strategy equilibria featuring finite numbers of active liquidity suppliers. None of the equilibria satisfies the competitive outcome, not even on average. Considering a sequence of equilibria with the number of active liquidity suppliers becoming large, we show that the aggregate stochastic price schedule converges to the deterministic competitive price schedule. We also provide a benevolent rationale for the frequent order cancellations that occur in limit order book markets.

Keywords: imperfect competition in schedules, financial markets, limit orders, tail condition, flickering quotes, fleeting orders, fill ratio
1 Introduction

Limit order book markets have all but replaced traditional exchanges. Accordingly, this market structure is of great interest to traders, regulators, and academics. A workhorse for studying limit order book markets is the conditional tail expectation condition (Rock 1990, Glosten 1994); in a market in which quoters face informed trade, this tail condition is a zero-profit condition which states that limit prices equal the conditional expected value of the asset, given the execution of the limit orders. Hereafter, we refer to a price schedule that satisfies the tail condition as a competitive price schedule.

This paper makes three contributions to the literature. The first is a negative result. In a static model, we demonstrate that having infinitely many liquidity suppliers is not sufficient to invoke the competitive price schedule. What underlies our result is that the equilibria we find (i) are not symmetric and (ii) are zero-rent. Specifically, in each of the equilibria, infinitely many liquidity suppliers find it suboptimal to supply liquidity, whereas those who do supply liquidity break even. There are no barriers to entry in our model, the liquidity suppliers are identical, and the equilibrium notion is a standard static Nash equilibrium in schedules.

The equilibria we find are in mixed strategies, implying that one equilibrium of our static game repeated is a repetition of the mixed strategy. That is, active liquidity suppliers will cancel old quotes and replace them with new, randomly chosen, quotes. The frequent revision of quotes is exactly what occurs in limit order book markets, as orders are canceled.

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1 There is a literature that studies the limit order book market while abstracting from information asymmetry. In this literature, the tail condition has no role. Important papers include those by Seppi (1997), Parlour (1998), Foucault (1999), Parlour and Seppi (2003), Foucault, Kadan, and Kandel (2005), and Roşu (2009).

2 We posit that there is no equilibrium with infinitely many liquidity suppliers in which any supplier extracts rents.
and replaced in milliseconds.\textsuperscript{3}

The second contribution of this paper is to show that such aggressive order management is a benign feature of an equilibrium, and not necessarily evidence of manipulative intent. This phenomenon of frequent order cancellation—also known as flickering quotes, fleeting orders, or a high message-to-fill ratio—cannot be explained by the competitive price schedule.

Whereas having infinitely many liquidity suppliers is thus not sufficient to assume the tail condition and, furthermore, the tail condition is at odds with the flickering quotes phenomenon, the third contribution of this paper is a positive result. We prove the existence of a sequence of Nash equilibria for which the tail condition emerges at the limit. That is, the equilibrium random price schedule gets arbitrarily close to the competitive price schedule. Thus, the analytically tractable tail condition can nevertheless serve as an approximation for prices in limit order markets.

Our model is an example of a game with discontinuous payoffs. Here, the discontinuity in payoffs is due to the discriminatory nature of limit order markets: Offering shares at a price marginally lower than the opponent’s offering price is dramatically different from offering the shares at a price marginally higher, because the former implies a greater chance of trading with uninformed noise traders. Many games with discontinuous payoffs fail to possess equilibria in pure strategies. To see how an equilibrium in pure strategies can fail to exist in our model, consider a liquidation value that has unbounded support, and assume that the noise demand has the least upper bound $Z$. In a pure-strategy equilibrium, no more than $Z$ shares can be offered. A profitable deviation for anyone who offers shares is to offer those shares at infinity. This deviation generates unbounded expected profits. Clearly, this argument is an artifact of the assumption that noise traders are willing to pay any price.

\textsuperscript{3}Hasbrouck (2015) shows that quotes are astoundingly volatile: the number of quote changes is, on average, 20 times higher than the number of transactions, and the volatility implied by quote changes of less than 50 milliseconds is 5 times that implied by quote changes of 16 minutes.
However, this is the assumption that the literature favors because a no-trade equilibrium is avoided: When noise traders’ demands are price inelastic, liquidity suppliers can always recoup their losses to informed traders.\footnote{This is by far the most common setup in the literature. Among the many papers that use this environment are those by Glosten and Milgrom (1985), Diamond and Verrecchia (1987), Easley and O’hara (1987), and Colliard (2016).}

Interestingly, the equilibria we study here share similarities with those found in the Bertrand–Edgeworth literature. Specifically, in that literature, producers choose capacities in pure strategies and randomize the prices (Tirole 1988). Although in our model, competition is in schedules, in the equilibria we find, active liquidity suppliers choose deterministic quantities and randomize prices. Our equilibria are also similar to those in the Bertrand–Edgeworth literature in that the entire demand may not be satisfied at low prices.

An important goal of the Bertrand–Edgeworth literature is to provide a strategic foundation for competitive pricing. Specifically, Allen and Hellwig (1986a), Allen and Hellwig (1986b), and Vives (1986) demonstrate the convergence of mixed-strategy equilibria, as the number of suppliers becomes large, to the competitive deterministic price. Our paper extends this literature to a case in which competition is in schedules; the demand is random; suppliers face adverse selection; and the limit is a deterministic competitive schedule, rather than a single competitive price.

Back and Baruch (2013) and Biais, Martimort, and Rochet (2013) also model strategic liquidity provision in limit order markets, but they focus on economic environments in which equilibria in pure strategies exist. Dennert (1993) models a dealers’ market in an environment similar to ours, and finds the equilibrium in mixed strategies. In that paper, however, dealers compete in prices (quantities are exogenous). Kyle (1989), Vives (2011), and Rostek and Weretka (2012) also study competition in schedules with private information. However, these papers focus on a trading platform that uses a single price to clear the market. Although
many financial markets use a single price auction to open (and close) the trading day, we focus on a limit order market.

The rest of this paper is organized as follows. In Section 2 we describe our model and define the tail condition. In Section 3 we define the equilibrium with infinitely many liquidity suppliers. In Section 4 we demonstrate the existence of a sequence of Nash equilibria. In Section 5 we prove a convergence result. In Section 6 we provide some examples. In Section 7 we show how the model can be extended. In Section 8 we model a repeated game, and in Section 9 we conclude.

2 The Limit Order Book and the Tail Condition

Orders are anonymous, so the only attribute a market order carries is its size. Price priority means that the execution of a limit order placed away from the market occurs only when the size of the incoming marketable order is sufficiently large to walk through the book until it hits the limit order. Therefore, in the tail condition, the condition is that the size of the incoming marketable order must be larger than the number of shares quoted at the limit order or better (lower in the case of an offer, higher in the case of a bid).

Two aspects of the operation of the limit order market lead to this condition. First, as noted above, the market observes price priority. Second, limit orders are firm in that they cannot be canceled in the midst of a marketable order execution. Expressed probabilistically, the tail condition is

\[
P(q) = \begin{cases} 
E[\bar{v}|q \leq \bar{q}] & \text{offer side} \\
E[\bar{v}|q \geq \bar{q}] & \text{bid side}
\end{cases}
\]  

(1)

where \(\bar{v}\) is the liquidation value of the asset, \(\bar{q}\) is the size of the incoming market order, and \(P(\cdot)\) is the marginal price schedule. For \(q > 0\) \((q < 0)\), \(P(q)\) is the asking price (bidding price) for the \(q\)th unit. The tail condition is a zero-expected-profit condition.
Equation (1) gives the impression of being an explicit definition of the competitive price schedule, but the condition is actually implicit. Both \( P(\cdot) \) and \( \tilde{q} \) are to be determined jointly. Moreover, for \( P(\cdot) \) to be a price schedule, it must be a monotonically increasing function. But this can be so only if \( \tilde{q} \) and \( \tilde{v} \) are positively correlated, which again brings us back to the point that the right-hand side of (1) is endogenous.

The goal of this section is to define the economic environment in this paper, as well as to re-express the tail condition in a manner that can facilitate the calculus in this paper—that is, to express the competitive price schedule as a function of the exogenous parameters of the model.

We consider a market for a single risky asset. The market is organized as a pure limit order book with the usual price priority: An incoming marketable order walks the book, picking off outstanding limit orders at their limit prices. The timing is as follows: First, uninformed risk-neutral liquidity suppliers simultaneously submit limit orders. Second, a single liquidity demander submits a marketable order. Finally, the asset liquidates and pays the liquidation value \( \tilde{v} \). The liquidation value is drawn from either a discrete or a continuous distribution, bounded or unbounded. However, we require \( \tilde{v} \) to have a finite expected value, and we denote by \( V \) the least upper bound of \( \tilde{v} \).

We assume that any real number is a feasible price. We focus on equilibria in which buy and sell limit orders of liquidity suppliers do not cross—in other words, uninformed liquidity suppliers do not trade among themselves. In such equilibria, we can study the offer side separately from the bid side; hereafter, for simplicity of exposition, we focus on the offer side of the book.

The liquidity demander can be one of two types: a news trader (with probability \( \mu \)) or a

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5 A marketable order is an order that can be executed upon submission. Any buy (sell) limit order with a price higher (lower) than the ask price (bid price) is marketable. In particular, market orders are marketable. 6 If \( \tilde{v} \) is unbounded, then \( V \) is infinity.
noise trader (with probability $1 - \mu$). A news trader knows $\hat{v}$ and picks off all offers at prices lower than $\hat{v}$. A noise trader submits a market order of size $\hat{z}$. We assume that $\hat{z}$ is a finite integer (i.e., a number of whole lots) random variable.\(^8\) If $\hat{z} > 0$, then the order is a buy order. Let $Z^+$ denote the strictly positive part of the support, and let $Z$ be the maximal element in $Z^+$. We can trivially write\(^9\)

$$Z^+ \subset \{i : i \in \mathbb{N}\}$$

In addition, we require that $\hat{z}$ and $\hat{v}$ be independent. Note that we do not specify the distributions of $\hat{z}$ and $\hat{v}$, except in specific examples. The results given in this paper do not depend on the specifics of the distributions beyond what we have already assumed above. We do assume that those distributions, whatever they are, are common knowledge.

Next, we turn to the liquidity suppliers. We represent a countable collection of offers using a marginal price schedule (hereafter, \textit{price schedule}). Formally, a price schedule is a nondecreasing left continuous step function $P : (0, Q] \rightarrow \mathbb{R}$, with the interpretation that in total $Q$ units are offered, and the $q$th unit is offered at $P(q)$.\(^{10}\) The range of the price schedule corresponds to the set of limit prices, and the lengths of the intervals of consistency (i.e., flats) correspond to the sizes of the limit orders. If $P : (0, Q] \rightarrow \mathbb{R}$ stands for the entire collection of outstanding offers in the book, and $\hat{q}$ stands for the size of the incoming order, then

\[ \forall q \in (0, Q], \quad \{q \leq \hat{q}\} = \{q \leq \hat{q}, \text{demander is news}\} \cup \{q \leq \hat{q}, \text{demander is noise}\} \]

\[ = (\{\text{demander is news}\} \cap \{P(q) < \hat{v}\}) \cup (\{\text{demander is noise}\} \cap \{q \leq \hat{z}\}) \]

\(^7\)The news trader also picks off all bids at prices higher than $\hat{v}$, but because we focus on the offer side, this does not concern us.

\(^8\)In Section 7 we relax the assumption that $\hat{z}$ is an integer random variable.

\(^9\)If $\hat{z} < 0$, then the order is a sell order. Because we focus on the offer side, the possible negative values of $\hat{z}$ do not concern us.

\(^{10}\)Typically, one takes the domain as given, and considers functions on that domain. Here, the domain is part of the definition of a price schedule. Specifically, we restrict our attention to domains that are intervals with the left endpoint being zero but with the right endpoint, $Q$, being part of the definition of the price schedule.
The four events are disjoint, and we can fully characterize the distribution of $\tilde{q}$ conditional on $P(\cdot)$ via\(^{11}\)

\[
\text{Prob}(q \leq \tilde{q}) = \mu \text{Prob}(P(q) < \tilde{v}) + (1 - \mu) \text{Prob}(q \leq \tilde{z})
\]

The aggregate expected profits of the liquidity suppliers are

\[
E \int_{0}^{\infty} I\{q \leq \tilde{q}\} (P(q) - \tilde{v}) dq = \int_{0}^{\infty} \mu EI\{P(q) < \tilde{v}\} (P(q) - \tilde{v}) + (1 - \mu) EI\{q \leq \tilde{z}\} (P(q) - \tilde{v}) dq = \int_{0}^{\infty} \alpha(P(q)) + \beta(P(q)) \text{Prob} (q \leq \tilde{z}) d q
\]

where

\[
\alpha(p) := \mu EI\{p < \tilde{v}\} (p - \tilde{v})
\]

\[
\beta(p) := (1 - \mu)(p - E\tilde{v})
\]

The term $\alpha(P(q))$ represents the expected gains from trading with an informed liquidity demander (these gains are negative). The term $\beta(P(q)) \text{Prob} (\tilde{z} \geq q)$ represents the expected gains from trading with an uninformed liquidity demander.

A price schedule is referred to as competitive, and denoted $P^c(\cdot)$, if its domain is $(0, Z]$ and the integrand is identically zero:

\[
\forall q \in (0, Z], \quad \alpha(P^c(q)) + \beta(P^c(q)) \text{Prob} (q \leq \tilde{z}) = 0
\]

Thanks to the continuity and monotonicity of $\alpha(\cdot)$ and $\beta(\cdot)$, the existence and uniqueness of the competitive price schedule are guaranteed. Moreover, the set of discontinuity points of $P^c(\cdot)$ is $Z^+ \setminus \{Z\}$. This property is inherited from the term $\text{Prob}(q \leq \tilde{z})$. Figure 1 shows an example of a competitive price schedule.

We let $\text{Ask}^c = \lim_{q \searrow 0} P^c(q)$ be the lowest price at which units are offered. Then, $\text{Ask}^c$ satisfies

\[
\alpha(\text{Ask}^c) + \beta(\text{Ask}^c) \text{Prob}(0 < \tilde{z}) = 0
\]

\(^{11}\)In other words, we define the distribution of $\tilde{q}$ by means of the left continuous version of its survival function.
Figure 1: Graph of a competitive price schedule, $P^c(\cdot)$. In this example, $Z = 8$, and $Z^+ = \{3, 7, 8\}$. There are three offering prices, each of which corresponds to an element of $Z^+$. All offering prices are strictly higher than $E\tilde{v}$ and strictly lower than $V$.

A rearrangement of (4) yields the upper tail condition:

$$\forall q \in (0, Z], \quad \left( P^c(q) - E[\tilde{v} \mid q \leq \tilde{q}] \right) \text{Prob}(q \leq \tilde{q}) = 0$$

Hereafter, we refer to (4) as the tail condition.

It is apparent that a Nash equilibrium cannot satisfy the tail condition: A profitable deviation for any strategic liquidity supplier is to offer the shares at a slightly higher price (Bernhardt and Hughson 1997).

3 Definition of $(n, \Delta, M(\cdot))$ Equilibrium

We assume that there are infinitely many potential liquidity suppliers. Each of the potential liquidity suppliers is strategic and can submit a countable collection of limit orders. The collection of limit orders of the $i$th potential liquidity supplier is represented in terms of an individual price schedule $P_i : (0, Q_i] \to R$. Because each element in the range of $P_i(\cdot)$ represents a limit price, the cardinality of the range is countable.

We search for equilibrium in which the “first” $n + 1$ liquidity suppliers offer shares and the remaining liquidity suppliers abstain. We refer to the $n + 1$ active suppliers of liquidity...
as quoters. In the equilibrium we search for, each quoter posts a single offer of size $\Delta$ at a random price. The random prices are continuously distributed with the common distribution function $M(\cdot)$. In particular, the equilibrium is described by a triple $(n, \Delta, M(\cdot))$.

Instead of computing the possible joint payoffs for each possible profile of individual price schedules, we only define payoffs of a potential supplier, quoter or nonquoter, under the assumption that the remaining suppliers follow their conjectured strategy.\footnote{In the same way that one does not need to spell out the payoff of cooperation to show that defection is a Nash equilibrium in the prisoner’s dilemma.}

The conjectured equilibrium is not symmetric: $n + 1$ suppliers quote, while the remaining potential suppliers abstain. However, in the conjectured equilibrium, the problem a quoter faces and the problem a nonquoter faces differ in only one aspect: Whereas a quoter faces a book populated with $n$ random limit orders, a nonquoter faces a book populated with $n + 1$ random limit orders.

We use the $(n+1)$th potential liquidity supplier to represent a typical quoter. We consider the possibility that the $(n + 1)$th quoter employs an arbitrary price schedule, $P_{n+1} : (0, Q_{n+1}] \to R$, instead of the conjectured strategy, while all others employ their conjectured strategy. Let $\{\tilde{p}_j\}_{j=1}^n$ be the $n$ random prices the first $n$ quoters use, and define the supply function

$$
\tilde{S}_n(p) = \Delta \sum_{j=1}^n I\{\tilde{p}_j \leq p\}
$$

Then, $\tilde{S}_n(p)$ is the number of units offered by the first $n$ quoters at prices lower than or equal to $p$.

Because $P_{n+1}(\cdot)$ is arbitrary, we need to consider the possibility of a tie. A tie occurs when (i) at least two traders offer shares at a price $p$ and (ii) the incoming market order is large enough to pick off offers at $p$, but not large enough to pick off all offers at $p$.

Because a news trader sweeps all offers at prices lower than $\tilde{v}$, there is no tie when the
liquidity demander is a news trader. By contrast, there can be a tie when the liquidity demander is a noise trader. Let \( \tilde{q}_{n+1} \) be the number of units that the \((n+1)\)th quoter sells when employing \( P_{n+1}(\cdot) \). Price priority implies

\[
\{ q \leq \tilde{q}_{n+1} | \text{demander is noise} \} \supseteq \left\{ q + \tilde{S}_n(P_{n+1}(q)) \leq \tilde{z} \mid \text{demander is noise} \right\}
\]

The set inequality can be a strict inequality only if there is a tie at \( P_{n+1}(q) \). However, the range of \( P_{n+1}(\cdot) \) is countable (i.e., the suppliers can post, at most, countably many limit orders), and the \( n \) random prices are continuously distributed; therefore, the probability of a tie is zero. Thus, without spelling out how ties are broken, we know that

\[
\{ q \leq \tilde{q}_{n+1} | \text{demander is noise} \} = \left\{ q + \tilde{S}_n(P_{n+1}(q)) \leq \tilde{z} \mid \text{demander is noise} \right\} \text{ almost surely}
\]

We can now fully characterize the distribution of \( \tilde{q}_{n+1} \) conditional on \( P_{n+1}(\cdot) \), without reference to tie-breaking rules, via

\[
\text{Prob}(q \leq \tilde{q}_{n+1}) = \mu \text{Prob}(P_{n+1}(q) < \tilde{v}) + (1 - \mu) \text{Prob}\left(q + \tilde{S}_n(P_{n+1}(q)) \leq \tilde{z}\right)
\]

The expected payoff associated with \( P_{n+1}(\cdot) \) is

\[
E \int_0^{\tilde{q}_{n+1}} (P_{n+1}(q) - \tilde{v}) \, dq
= E \int_0^{\infty} I\{q \leq \tilde{q}_{n+1}\} (P_{n+1}(q) - \tilde{v}) \, dq
= \int_0^{\infty} \mu EI\{P_{n+1}(q) < \tilde{v}\} (P_{n+1}(q) - \tilde{v}) \, dq
+ (1 - \mu) \text{Prob}\left(q + \tilde{S}_n(P_{n+1}(q)) \leq \tilde{z}\right) (P_{n+1}(q) - \tilde{v}) \, dq
= \int_0^{\infty} \alpha(P_{n+1}(q)) + \beta(P_{n+1}(q)) \text{Prob}\left(q + \tilde{S}_n(P_{n+1}(q)) \leq \tilde{z}\right) \, dq \quad (7)
\]

Next, we want to compute the expected payoff of a nonquoter. We use the \((n+2)\)th potential liquidity supplier to represent an arbitrary nonquoter. We consider the possibility that the nonquoter employs an arbitrary price schedule \( P_{n+2} : [0, Q_{n+2}] \to R \), while all others employ
their conjectured strategy. The expected payoff associated with \( P_{n+2}(\cdot) \) is

\[
\int_0^\infty \alpha(P_{n+2}(q)) + \beta(P_{n+2}(q)) \text{Prob}\left(q + \Delta \sum_{j=1}^{n+1} I\{\tilde{p}_j \leq P_{n+2}(q)\} \leq \tilde{z}\right) dq
\]  

(8)

The only difference between (8) and (7) is that a nonquoter faces a book populated with \( n + 1 \) random limit orders.

**Definition 1.** The triple \((n, \Delta, M(\cdot))\) is a Nash equilibrium if:

1. The maximum of (7) over the class of all price schedules is attained at any price schedule with domain \((0, \Delta]\) and a singleton range \(\{p\}\) for which \(p\) is in the support of \(M(\cdot)\). In other words,

\[
P_{n+1}(q) = \begin{cases} p & q \in (0, \Delta] \\ \text{undefined} & \text{otherwise} \end{cases}, \quad p \in \text{support of } M(\cdot)
\]  

(9)

2. The maximum of (8) over the class of all price schedules is zero.

The first condition in the definition of equilibrium implies that a quoter finds it optimal to submit a single limit order, of size \(\Delta\), at a random price distributed according to \(M(\cdot)\). The second condition in the definition implies that a nonquoter finds it optimal to abstain.

We reiterate that in equilibrium, the number of quoters and the collection of limit orders each of quoters submits is endogenous. In this section, we have narrowed our attention to equilibria in which the endogenous number of quoters is \(n + 1\), the endogenous number of limit orders each quoter submits is one, the endogenous size of those limit orders is \(\Delta\), and the endogenous distribution of the limit prices is \(M(\cdot)\). We have not yet shown that there is an equilibrium that can be described by the triple \((n, \Delta, M(\cdot))\). In the next section we show that in fact we can find many equilibria.
4 Sequence of Nash Equilibria

In this section we construct a sequence of zero-rent equilibria, \((n_m, \Delta_m, M_m(\cdot))\), parameterized by \(m\). In comparison with the competitive price schedule, the Nash price schedules differ in the following dimensions. Whereas under \(P^c(\cdot)\) \(Z\) units are offered in total, in the \(m\)th Nash equilibrium \(Z + 1/m\) units are offered. And whereas under \(P^c(\cdot)\) the best offering price is \(\text{Ask}^c\), in each of the Nash equilibria the best offering price is (with probability one) strictly greater than \(\text{Ask}^c\). Thus, the Nash equilibria do not satisfy the tail condition, nor do they satisfy the tail condition “on average.”

We construct the sequence as follows. For every \(m \in \mathbb{N}\), we define

\[
\begin{align*}
n_m &\equiv mZ \\
\Delta_m &\equiv 1/m
\end{align*}
\]

(10)

Next, we define the support of \(M_m(\cdot)\) to be the interval \([\text{Ask}^c, V]\). Note that the support of the mixing distribution does not depend on \(m\). In its support, we define \(M_m(\cdot)\) implicitly as a solution of the functional equation

\[
\forall p \in [\text{Ask}^c, V], \quad \alpha(p) + \beta(p) \sum_{z \in \mathbb{Z}^+} \text{Prob}(z = \tilde{z})B(mz - 1; mZ, M_m(p)) = 0
\]

(11)

where \(B(k; n, h)\) is the distribution function of a binomial random variable with parameters \(n\) and \(h\).\(^{13}\) We need to show that \(M_m(\cdot)\) is well defined. In other words, we need to show that there is a solution to the functional equation (11) and, moreover, that the solution is a distribution function. This is the goal of the following lemma, the proof of which is in the Appendix.

Lemma 4.1. There exists a distribution function \(M_m(\cdot)\) that satisfies (11). It is continuous and strictly increasing on its support \([\text{Ask}^c, V]\).

\(^{13}\)\(B(k; n, h)\) is the probability that out of \(n\) independent Bernoulli trials, each with probability of success \(h\), there are at most \(k\) successes.
Admittedly, the construction of \((n_m, \Delta_m, M_m(\cdot))\) is opaque, but nevertheless, in Theorem 4.3, below, we prove that \((n_m, \Delta_m, M_m(\cdot))\) is a zero-rent Nash equilibrium.

To develop the intuition, note that construction (10) implies that the cumulative number of units offered by the first \(n_m\) quoters at prices equal to or lower than \(p\) is

\[
\tilde{S}_{n_m}(p) = \frac{1}{m} \sum_{i=1}^{mZ} I\{\tilde{p}_i \leq p\}
\]

Thus, the support of \(\tilde{S}_{n_m}(p)\) is

\[
\left\{ \frac{0}{m}, \frac{1}{m}, \cdots, \frac{m-1}{m}, \frac{m+1}{m}, \cdots, \frac{2m-1}{m}, \frac{2m}{m}, \frac{2m+1}{m}, \cdots, \frac{Zm-1}{m}, \frac{Zm}{m} \right\}
\]

(12)

Because the size of a noise order, \(\tilde{z}\), is an integer bounded by \(Z\), if the liquidity demander is a noise trader, then

\[
\tilde{S}_{n_m}(p) < \tilde{z} \iff \tilde{S}_{n_m}(p) + \frac{1}{m} \leq \tilde{z}
\]

In other words, in equilibrium, the \((n+1)\)th quoter offers a block of \(1/m\) units at some price \(p \in [\text{Ask}^c, V]\). If the liquidity demander is a noise trader, the quoter either gets to sell the entire block of \(1/m\) units or does not get to trade at all. The former has the probability

\[
\text{Prob}(\tilde{S}_{n_m}(p) < \tilde{z})
\]

To reveal the economic meaning of the mixing distribution \(M_m(\cdot)\), we reexpress (11).

**Lemma 4.2.**

\[
\forall p \in [\text{Ask}^c, V], \quad \alpha(p) + \beta(p) \text{Prob}(\tilde{S}_{n_m}(p) < \tilde{z}) = 0
\]

(13)

The proof of the lemma is in the Appendix. We refer to (13) as the *Nash condition*. The Nash condition ensures that the \((n+1)\)th quoter earns zero expected profit, no matter at what price in the support of the equilibrium mixing strategy the \(1/m\) units are offered.

In other words, the Nash condition ensures that, when the \((n+1)\)th quoter follows the

\footnote{The condition is necessary but not sufficient for equilibrium because we still need to show that there are no profitable deviations.}
conjectured equilibrium strategy, the expected gains from trading with an informed liquidity demander, $\alpha(p)$, and the expected gains from trading with a noise liquidity demander, $\beta(p) \text{Prob}(\bar{S}_{nm}(p) < \bar{z})$, add up to zero.

**Theorem 4.3.** The triple $(n_m, \Delta_m, M_m(\cdot))$ is a Nash equilibrium. Furthermore, it is a zero-rent Nash equilibrium.

The proof is in the Appendix. The presence of the inactive liquidity suppliers is not necessary to support the equilibrium. The active traders’ strategies also form an equilibrium in an economy consisting of $1 + n_m$ potential liquidity suppliers, all of whom are quoting. By contrast, Theorem 4.3 demonstrates that even with infinitely many potential suppliers, the equilibrium price schedule differs from the competitive price schedule.

Before we close this section, let us consider an example. Assume that the log of the liquidation value is a standard normal random variable. In particular, $E\bar{v} = \exp(1/2)$ and $V = \infty$. The noise order demand takes one of four values, $\{-2, -1, 1, 2\}$ with equal probabilities, in particular $Z = 2$. We set $\mu = 1/3$. Thus,

$$
\alpha(p) = \frac{1}{3} \int_p^\infty (p - v) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{[\ln(v)]^2}{2} \right) dv
$$

$$
\beta(p) = \frac{2}{3} (p - \exp(1/2))
$$

In the competitive equilibrium, two units are offered. The first unit is offered at a price that solves the equation

$$
\alpha(p) + \beta(p) \text{Prob}(1 \leq \bar{z})
$$

This price is also Ask$^c$. The second unit is offered at a price that solves the equation

$$
\alpha(p) + \beta(p) \text{Prob}(2 \leq \bar{z})
$$

We compute the Nash equilibrium that corresponds to $m = 1$, so there are three quoters, each of whom offers one lot. In Example 2 of Section 6, we consider this example in detail.
and show that the mixing distribution is
\[ p \in [\text{Ask}^0 \infty), \quad 1 + \frac{\alpha(p)}{\beta(p) \text{Prob}(\tilde{z} > 0)} \]

Let \( \tilde{p}\) denote the \( j \)th order statistics of the random prices. Thus, \( \tilde{p} \) is the price at which the \( j \)th lot is offered. We use a computer algebra system to compute the entries in the following table of offering prices (the code is in the Appendix).

<table>
<thead>
<tr>
<th>Lot</th>
<th>Competitive Offering Price</th>
<th>Nash Equilibrium Offering Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2.148894561</td>
<td>Expected 99% Confidence Interval</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.320657094</td>
</tr>
<tr>
<td>2nd</td>
<td>2.506363216</td>
<td>2.670879942</td>
</tr>
<tr>
<td>3rd</td>
<td>NA</td>
<td>3.837032686</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[2.149578757, 3.411256976]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[2.166449935, 5.434745336]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[2.308004380, 14.84086766]</td>
</tr>
</tbody>
</table>

5 Convergence

In this section, endowed with the sequence of equilibria, we prove a convergence result. In the main theorem of this section, Theorem 5.3, we show that for all \( q \notin \mathcal{Z}^+ \), the sequence of random prices at which the \( q \)th unit is offered at the \( m \)th equilibria converges in distribution to \( p^e(q) \).

Denote by \( \tilde{p}_{m,j} \) the random price of the \( j \)th quoter in the \( m \)th equilibria. Classic convergence theorems in probability theory deal with the averages of independent and identically distributed (i.i.d.) random variables in a sequence. But in this paper we deal with a triangular array of random prices:15

\[
\begin{align*}
\tilde{p}_{1,1} & \quad \ldots & \quad \tilde{p}_{1,Z} & \quad \tilde{p}_{1,j} & \sim M_1(\cdot) \\
\tilde{p}_{2,1} & \quad \ldots & \quad \tilde{p}_{2,Z} & \quad \tilde{p}_{2,2} & \sim M_2(\cdot) \\
\vdots & \quad \vdots & \quad \vdots & \quad \ddots & \quad \vdots \\
\tilde{p}_{m,1} & \quad \ldots & \quad \tilde{p}_{m,Z} & \quad \tilde{p}_{m,2} & \sim M_m(\cdot) \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \ddots & \quad \vdots \\
\end{align*}
\]

15Initially, we neglect the limit order of the \((n_m + 1)\)th quoter. We do count it in our main theorem, Theorem 5.3.
The classic convergence results apply to the limiting distribution of the row average. We face two difficulties that need to be overcome. The first difficulty is that a priori we do not know that a limiting distribution exists. The second difficulty is that we ultimately need to prove that the price schedule converges to $P^c(\cdot)$. That is, we need to prove results that go beyond the row averages of the random prices. So, instead of considering the triangular array of random prices, we consider a continuum of triangular arrays of Bernoulli random variables, parameterized by $p$:

$$I\{\tilde{p}_{1,1} \leq p\} \ldots I\{\tilde{p}_{1,Z} \leq p\} \quad \tilde{p}_{1,j} \sim M_1(\cdot)$$

$$I\{\tilde{p}_{2,1} \leq p\} \ldots I\{\tilde{p}_{2,Z} \leq p\} \ldots I\{\tilde{p}_{2,2Z} \leq p\} \quad \tilde{p}_{2,j} \sim M_2(\cdot)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots$$

$$I\{\tilde{p}_{m,1} \leq p\} \ldots I\{\tilde{p}_{m,Z} \leq p\} \ldots I\{\tilde{p}_{m,2Z} \leq p\} \ldots I\{\tilde{p}_{m,mZ} \leq p\} \quad \tilde{p}_{m,j} \sim M_m(\cdot)$$

For a given $p$, a row average of the above triangular array should converge to $\lim M_m(p)$, provided the limit exists. Thus, we introduce the row average function that maps a real number $p$ to the average of the $m$th row of the triangular array associated with $p$:

$$\tilde{M}_m(p) \overset{d}{=} \frac{1}{mZ} \sum_{i=1}^{mZ} I\{\tilde{p}_{m,i} \leq p\}, \quad \tilde{p}_i \text{ i.i.d. } M_m(\cdot) \quad (14)$$

where the equality is in distribution. For a given sample of the $mZ$ random prices, $\tilde{M}_m(\cdot)$ is known as the empirical distribution function associated with the sample, and $\tilde{M}_m(p)$ is the sample mean of the $mZ$ Bernoulli trials, $I\{\tilde{p}_{m,i} \leq p\}$, each with probability of success $M_m(p)$.

Note that the supply function of the first $n_m = mZ$ quoters is related to $\tilde{M}_m(\cdot)$ via

$$\tilde{S}_{n_m}(p) \overset{d}{=} Z\tilde{M}_m(p) \quad (15)$$

To prove the convergence results, we rely on (i) the tendency of sample means to converge to the limit of the probability of success and (ii) the similarity between the tail condition and the Nash condition.
In Lemma 5.1, we prove the variant of the law of large numbers we need. In Lemma 5.2, we use the similarity between the tail and Nash conditions to prove that $Z \tilde{M}_m(p)$ converges in distribution to an “inverse” of $P^c(\cdot)$; that is, if $P^c(\cdot)$ had an inverse, then this is exactly what Lemma 5.2 would imply. Theorem 5.3 delivers the convergence of the random price schedules in the sequence of Nash equilibria to $P^c(\cdot)$.

**Lemma 5.1.** Fix $p$. If $M_{m_k}(p)$ is a convergent subsequence, then $\tilde{M}_{m_k}(p)$ converges in distribution to the same limit.\(^\text{16}\) In other words,

$$\lim \text{Prob}(\tilde{M}_{m_k}(p) \leq x) = \begin{cases} 0 & \lim_{m_k \to \infty} M_{m_k}(p) > x \\ 1 & \lim_{m_k \to \infty} M_{m_k}(p) < x \end{cases}$$

The proof is in the Appendix. To prove the next lemma, we contrast the tail condition with the Nash condition.

**Lemma 5.2.** Fix $q \in (0, Z)$, with $q \notin Z^+$. We have

$$\lim \text{Prob}(\tilde{M}_m(p) \leq q/Z) = \begin{cases} 0 & p > P^c(q) \\ 1 & p < P^c(q) \end{cases}$$

The proof is in the Appendix.

\(^{16}\)We prove that the subsequence of random variables $\tilde{M}_{m_k}(p)$ converges in distribution to the constant $\lim M_{m_k}(p)$. Recall that any constant $c$ is a degenerate random variable with distribution

$$\text{Prob}(c \leq x) = \begin{cases} 0 & c > x \\ 1 & c \leq x \end{cases}$$

Thus, a sequence of random variables $\tilde{x}_n$ converges in distribution to a constant $c$ if

$$\lim \text{Prob}(\tilde{x}_n \leq x) = \begin{cases} 0 & c > x \\ 1 & c > c \end{cases}$$

Convergence at $x = c$ is not required, because $c$ is a discontinuity point of the limiting distribution. For example, the sequence of degenerate random variables (i.e., real numbers)

$$x_n = c + \frac{(-1)^n}{n}$$

converges in distribution to $c$ even though the sequence $\text{Prob}(x_n \leq c)$ oscillates between zero and one, and therefore does not have a limit.
Let $\tilde{P}^m(\cdot)$ denote the random aggregate price schedule in the $m$th equilibrium that results from the entire collection of $n_m + 1$ limit orders. For each $m$, we have \[
{\tilde{P}^m(q) \leq p} \quad \| \quad \left\{ \sum_{i=1}^{n_m} \Delta_m I_{\tilde{p}_m,i \leq p} \geq q \right\} \subseteq \left\{ \sum_{i=1}^{1+n_m} \Delta_m I_{\tilde{p}_m,i \leq p} \geq q \right\} \subseteq \left\{ \sum_{i=1}^{n_m} \Delta_m I_{\tilde{p}_m,i \leq p} + \Delta_m \geq q \right\} \quad \| \quad \left\{ \tilde{S}_{\tilde{z}_m}(p) \geq q \right\} \quad \| \quad \left\{ \tilde{S}_{\tilde{z}_m}(p) \geq q - \frac{1}{m} \right\} \]
Thus, from (15) \[
Prob(Z \tilde{M}_m(p) \geq q) \leq Prob(\tilde{P}^m(q) \leq p) \leq Prob(Z \tilde{M}_m(p) \geq q - \frac{1}{m}) \quad (16)
\]

**Theorem 5.3.** Fix $q \in (0, Z)$, with $q \notin Z^+$. Then $\tilde{P}^m(q)$ converges in distribution to $P^c(q)$.

In other words, \[
\lim_{m \to \infty} Prob(\tilde{P}^m(q) \leq p) = \begin{cases} 0 & p < P^c(q) \\ 1 & p > P^c(q) \end{cases}
\]

The proof is in the Appendix and is based on a sandwich argument applied to (16). Theorem 5.3 has nothing to say about $q > Z$. That said, in the competitive equilibrium a total of $Z$ units are offered, whereas in the Nash equilibrium the total is $Z + \Delta_m \to Z$.

6  **Examples**

In this section, we provide two examples in which it is possible to explicitly compute the mixing distribution.

**Example 1.** $Z^+$ is a singleton.

Because $Z$ is the only element in $Z^+$, $Prob(Z = \tilde{z}) = Prob(0 < \tilde{z})$, and \[
Prob(\tilde{S}_{\tilde{z}_m}(p) < \tilde{z}) = Prob(0 < \tilde{z}) \times Prob(\tilde{S}_{\tilde{z}_m}(p) < Z) = Prob(0 < \tilde{z}) \times (1 - Prob(\tilde{S}_{\tilde{z}_m}(p) = Z)) = Prob(0 < \tilde{z}) \times (1 - (M_m(p))^{mZ})
\]
Inserting this result into (13), we obtain
\[ \forall p \in (\text{Ask}^c, V), \quad M_m(p) = \left( 1 + \frac{\alpha(p)}{\beta(p) \text{Prob}(0 < \tilde{z})} \right)^{1/mZ} \]
where Ask\(^c\), as always, is given by (5).

**Example 2.** \( m = 1 \) and the conditional distribution of \( \tilde{z} \), given \( \{0 < \tilde{z}\} \), is \( \mathcal{U}\{1, 2, \ldots, Z\} \).

In this example, we consider only \( m = 1 \); thus, \( n_1 = Z \). Our goal is to compute \( M_1(\cdot) \) explicitly.

For each \( p \), the support of \( \tilde{S}_{n_1}(p) \) is \( \{0, 1, \ldots, Z\} \), so\(^{17} \)
\[ \sum_{i=1}^{Z} \text{Prob}(\tilde{S}_{n_1}(p) \geq i) = E\tilde{S}_{n_1}(p) = Z M_1(p) \]
Thus,
\[ \sum_{i=1}^{Z} \text{Prob}(\tilde{S}_{n_1}(p) < i) = Z(1 - M_1(p)) \]
Now, \( \tilde{z} \), given \( \{0 < \tilde{z}\} \), is uniformly distributed, so
\[ \text{Prob}(\tilde{S}_{n_1}(p) < \tilde{z}) = \sum_{z \in \tilde{z}^+} \text{Prob}(z = \tilde{z}) \text{Prob}(\tilde{S}_{n_1}(p) < z) \]
\[ = \frac{\text{Prob}(0 < \tilde{z})}{Z} \sum_{z=1}^{Z} \text{Prob}(\tilde{S}_{n_1}(p) < z) = \text{Prob}(0 < \tilde{z})(1 - M_1(p)) \]
where the first equality holds because \( \tilde{S}_{n_1}(p) \) is a nonnegative random variable.

Inserting the result into (13), we obtain
\[ \forall p \in (\text{Ask}^c, V), \quad M_1(p) = 1 + \frac{\alpha(p)}{\beta(p) \text{Prob}(0 < \tilde{z})} \]
where Ask\(^c\) is given by (5).

Of course, if \( Z = 1 \), then Example 2 is reduced to Example 1 with \( m = 1 \).

\(^{17}\text{Summation by parts implies that for any nonnegative integer random variable } \tilde{x}, \ E\tilde{x} = \sum_{i=1}^{\infty} \text{Prob}(\tilde{x} \geq i). \)
Here, we use this result with the random variable \( \tilde{S}_{n_1}(p) \).
7 An Extension

We relax the assumption that $\tilde{z}$ is an integer-valued random variable. Instead, we assume that $\tilde{z}$ can be embedded in a grid. Specifically, we replace property (2) with

$$\exists \tau_{ick} \in \mathbb{R}^+ \text{ such that } \mathcal{Z}^+ \subset \{i \times \tau_{ick} \mid i \in \mathbb{N}\}$$  \hspace{1cm} (2')

The assumption that $\tilde{z}$ is an integer random variable corresponds to the special case $\tau_{ick} = 1$.

The following adjustment needs to be made to accommodate the change: Replace (10) with

$$n_m \equiv mZ/\tau_{ick}$$
$$\Delta_m \equiv \tau_{ick}/m$$  \hspace{1cm} (10')

The rest of this paper (definitions, theorems, lemmas, and proofs) is unchanged.

Because every finite, continuously distributed random variable $\tilde{z}$ can be approximated using a discrete random variable, we can use the relaxed assumption (2') to conclude that a Nash equilibrium is also arbitrarily close to the competitive supply function when $\tilde{z}$ is continuously distributed.\footnote{This is analogous to the notion that a no-arbitrage argument applied in a binomial world can be used to approximate derivative securities in the continuous economy of Black and Scholes.}

We illustrate the convergence using the uniform example (Example 2). Assume we are interested in approximating a model in which the noise trader is equally likely to be a buyer or a seller and the conditional distribution of the noise order, given that the order is positive, is a continuous random variable with uniform distribution on the interval $(0, 1)$.

Solving the tail condition yields

$$\forall q < 0 \leq 1 \quad q = 1 + \frac{\alpha(P^c(q))}{\beta(P^c(q)) \times 1/2}$$

Let $S^c(\cdot) : \text{[Ask}^c, P^c(1)]$ be

$$S^c(p) = 1 + \frac{\alpha(p)}{\beta(p) \times 1/2}$$

Then, for $q \in (0, 1]$, $P^c(q)$ is the inverse of $S^c(p)$, and $S^c(p)$ is the total number of units offered in the competitive equilibrium at prices lower than or equal to $p$. 
Consider an approximation of the continuous standard uniform random variable: Set \( \tau_{ick} = 1/10^k \) for some large integer \( k \), and assume that \( \text{Prob}(0 < \tilde{z}) = 1/2 \) and that the conditional distribution of \( \tilde{z} \), given \( \{0 < \tilde{z}\} \), is

\[
U\{\tau_{ick}, 2\tau_{ick}, \ldots, 10^k \times \tau_{ick}\}
\]

Then, \( Z = 1, n_1 = m10^k, \) and \( \Delta_m = 1/(m10^k) \). When \( m = 1 \), we can compute the mixing distribution explicitly as in Example 2, above, yielding

\[
p \in [\text{Ask}^c, V], \quad 1 + \frac{\alpha(p)}{\beta(p) \times 1/2}
\]

which, for \( p \in [\text{Ask}^c, P^c(1)] \), is exactly \( S^c(p) \).

The total number of units offered in the Nash equilibrium is

\[
\frac{1}{10^k} \sum_{i=1}^{10^k+1} I\{\tilde{p}_i \leq p\} \approx \text{the mixing distribution} = S^c(p)
\]

where for the approximation we use the law of large numbers.

8 Flickering Quotes

In this section, we present a repeated game model of dynamic liquidity provision. This dynamic model generates an arbitrarily high message-to-fill ratio—where a message is any new order, order modification, or order cancellation.

The stage game is similar to the static game, but we make two adjustments. First, we add the possibility that a liquidity demander may not show up, and second, we assume that only at the end of the stage game does everyone get to see the limit orders initially submitted. We interpret the time the stage game lasts to be the latency of the exchange messaging network. Thus, the quotes will not be seen until the end of the stage game. The stage game and the
static game are equivalent, and both possess a mixed-strategy equilibrium.\textsuperscript{19} The repeated game ends as soon as the news trader trades. Otherwise, the stage game repeats.

A trivial Nash equilibrium of the repeated game is repeatedly playing the mixed-strategy equilibrium of the static game.

In a static game, randomization is used to avoid undercutting. A way to avoid undercutting in a transparent and dynamic game is to keep revising the limit orders. Thus, in the

\textsuperscript{19}To see the equivalence, we map the event \{liquidity demander is noise and demand is zero\} in the static game to the event \{no liquidity demander is present\} in the stage game.
repeated game, limit orders are revised despite the lack of trade. Put differently, the rationale for the frequent order cancellation in the model is the undoing of market transparency. Because liquidity suppliers break even, we in fact prove that frequent order cancellation is not necessarily a sign of fraudulent rent extraction.

9 Conclusion

The tail condition is a pillar of the literature of limit order markets. In this paper, we demonstrate that it is not sufficient to assume that there are infinitely many liquidity suppliers in order to invoke the condition. On a positive note, we show a sequence of Nash equilibria that does converge to the competitive outcome.

We model dynamic liquidity provision as a repeated game of our static model. One equilibrium of such a game is a repetition of the mixed-strategy one we describe. This mixing corresponds to order cancellations and replacements. Thus, flickering quotes can be a benign part of an equilibrium rather than the result of manipulative behavior.
Appendix

Proof of Lemma 4.1. Our goal is to show that we can define a continuous and strictly increasing distribution function, \( M_m(\cdot) \), with support \([\text{Ask}^c, V]\), that satisfies (11) in its support.

Define the function \( H : [0, 1] \rightarrow [0, \text{Prob}(0 < \tilde{z})] \) via

\[
H(h) = \sum_{z \in \mathbb{Z}^+} \text{Prob}(z = \tilde{z}) B(mz - 1; mZ, h)
\]

Then, \( H(\cdot) \) is continuous and strictly decreasing with \(^{20}\)

\[
H(0) = \text{Prob}(0 < \tilde{z})
\]
\[
H(1) = 0
\]

Therefore, \( H^{-1} : [0, \text{Prob}(0 < \tilde{z})] \rightarrow [0, 1] \) exists, it is continuous and strictly decreasing, and it satisfies the boundary conditions

\[
H^{-1}(\text{Prob}(0 < \tilde{z})) = 0
\]
\[
H^{-1}(0) = 1
\]

Set \( M_m : [\text{Ask}^c, V] \rightarrow [0, 1] \) via

\[
M_m(p) = H^{-1}\left(\frac{-\alpha(p)}{\beta(p)}\right)
\]

By construction, \( M_m(\cdot) \) satisfies (11).

We need to verify that \( M_m(\cdot) \) is a continuous and strictly increasing function with \( M_m(\text{Ask}^c) = 0 \) and \( M_m(V) = 1 \).

\(^{20}\)Informally, when we increase \( h \), the probability of success in each trial, the probability of having at most \( x \) strictly decreases. Formally, we write

\[
B(k; n, h) = (n - k) \binom{n}{k} \int_0^{1-h} t^{n-k-1}(1-t)^k dt
\]

and simply differentiate with regard to \( h \).
From the definition of $\text{Ask}^c$, we find that $-\alpha(\text{Ask}^c)/\beta(\text{Ask}^c) = \text{Prob}(0 < \tilde{z})$. Therefore, $M_m(\text{Ask}^c) = 0$. Because $\alpha(V) = 0$, we also have $M_m(V) = 1$.

The continuity of $H^{-1}(\cdot)$, $\alpha(\cdot)$, and $\beta(\cdot)$ and the fact that in $[\text{Ask}^c, V]$, $\beta(\cdot) > 0$ imply the continuity of $M_m(\cdot)$.

To see that $p \mapsto M_m(p)$ is strictly increasing, note that in $[\text{Ask}^c, V]$ the function $p \mapsto -\alpha(p)/\beta(p)$ is strictly decreasing. Because $H^{-1}(\cdot)$ is also strictly decreasing, it follows that $p \mapsto M_m(p)$ is strictly increasing.

\begin{proof}[Proof of Lemma 4.2] Let $M_m(\cdot)$ be a distribution function that satisfies (11). In Lemma 4.1 we prove that such a distribution function exists. Consider a sample of $mZ$ i.i.d. prices distributed according to the distribution function $M_m(\cdot)$.

We note that

$$\tilde{S}_{nm}(p)/Z = \frac{1}{mZ} \sum_{i=1}^{mZ} I\{\tilde{p}_i \leq p\}$$

is the proportion of successes in a sample of $mZ$ Bernoulli trials, each Bernulli trial has a probability of success $M_m(p)$.

Thus,

$$EI\{\tilde{S}_{nm}(p) < z\} = \text{Prob}(\tilde{S}(p) < z) = \text{Prob}(\tilde{S}(p)/Z < z/Z)$$

$$= \text{Prob (proportion of successes is strictly smaller than } z/Z)$$

$$= \text{Prob (count of successes is strictly smaller than } mz)$$

$$= B(mz - 1; mZ, M_m(p))$$

Integrating the first term over all possible realizations of $z$, we get

$$EEI\{\tilde{S}_{nm}(p) < \tilde{z}\} = \text{Prob}(\tilde{S}_{nm}(p) < \tilde{z})$$

25
Integrating the last term, we get

\[ EB(m\tilde{z} - 1; mZ, M_m(p)) = \sum_{z \in \mathbb{Z}^+} \text{Prob}(z = \tilde{z}) B(n_m z / Z - 1; n_m, M_m(p)) \]

where the equality holds because for negative values of \(\tilde{z}\), the term in the expectation is zero. Thus, we have proved that

\[ \text{Prob}(\tilde{S}_{nm}(p) < \tilde{z}) = \sum_{z \in \mathbb{Z}^+} \text{Prob}(z = \tilde{z}) B(n_m z / Z - 1; n_m, M_m(p)) \]

which is what we need to prove to show that (11) implies (13). \qed

**Proof of Theorem 4.3.** We first consider the problem of the quoter, prototyped by the \((n_m + 1)\)th quoter, assuming all other liquidity suppliers follow the equilibrium strategy. From (7), we know that the expected profits associated with an arbitrary price schedule \(P_{nm+1}(\cdot)\):

\[ \int_0^\infty \alpha(P_{nm+1}(q)) + \beta(P_{nm+1}(q)) \text{Prob}(q + \tilde{S}_n(P_{nm+1}(q)) \leq \tilde{z}) \, dq \]

Ignoring the constraint that \(q \to P_{nm+1}(q)\) must be nondecreasing, we can solve this optimization problem pointwise (for each \(q\), let \(P_{nm+1}(q)\) maximize the integrand). If an argument of maximum of the unconstraint problem is a nondecreasing price schedule, then it is also a solution to the constraint problem.

Therefore, for every \(q\), we want to find a value of \(p\) that maximizes the integrand

\[ \alpha(p) + \beta(p) \text{Prob}(q + \tilde{S}_{nm}(p) \leq \tilde{z}) \] (17)

Fix \(q \in (0, 1/m]\), and recall that the support of \(\tilde{S}_{nm}(p)\) (see (12)) is a grid with steps \(\Delta_m = 1/m\). Thus,

\[ \text{Prob}(q + \tilde{S}_n(p) \leq \tilde{z}) = \text{Prob}(\tilde{S}_n(p) < \tilde{z}) \]

Thus, for \(q \in (0, 1/m]\), the integrand (17) equals

\[ \alpha(p) + \beta(p) \text{Prob}(\tilde{S}_{nm}(p) < \tilde{z}) \begin{cases} \leq 0 & p < \text{Ask}^e \\ = 0 & p \in [\text{Ask}^e, V] \\ = 0 & p > V \end{cases} \]
The inequality for $p < \text{Ask}^c$ arises from the monotonicity of $\alpha(\cdot)$ and $\beta(\cdot)$ (i.e., even in the competitive equilibrium, units are not offered below $\text{Ask}^c$). The equality for $p$ in the support of the mixing distribution is the Nash condition (13). For $p > V$, the integrand is again zero because $\alpha(p) = 0$ for all $p \geq V$, and $\text{Prob}\left(\tilde{S}_{nm}(V) < \tilde{z}\right) = 0$.

We conclude that for each $q \in (0, 1/m]$, any $p \in [\text{Ask}^c, V]$ is optimal. We now consider pointwise maximization for $q > 1/m$. Because the integrand (17) decreases in $q$, the integrand cannot be strictly positive no matter which $p$ we plug in.

In summary, we have shown that the maximum expected profit is zero and that it is optimal to offer any $q$, provided $q \leq \Delta_m$, at any price in the support of $M_m(p)$. In particular, it is optimal to offer the entire “block” of $1/m$ at a random price with distribution $M_m(p)$.

We now turn our attention to the nonquoters, prototyped by the $(n + 2)$th traders. Equation 8 expresses the expected payoff associated with an arbitrary price schedule. Maximizing the integrand in (8) pointwise (i.e., ignoring the monotonicity requirement) means that for every $q > 0$, we need to find a $p$ that maximizes the integrand

$$\alpha(p) + \beta(p) \text{Prob}\left(q + \tilde{S}_{nm}(p) + \Delta I_{\{\tilde{p}_{n+1} \leq p\}} \leq \tilde{z}\right)$$

We have

$$\alpha(p) + \beta(p) \text{Prob}\left(q + \tilde{S}_{nm}(p) + \Delta I_{\{\tilde{p}_{n+1} \leq p\}} \leq \tilde{z}\right) \leq \alpha(p) + \beta(p) \text{Prob}\left(q + \tilde{S}_{nm}(p) \leq \tilde{z}\right) \leq 0$$

The last inequality is the first part of the proof in which we proved that the integrand (17) is nonpositive. In conclusion, we have shown that the maximum expected profit cannot be strictly positive, and hence it is optimal for the $(n + 2)$th liquidity supplier to abstain. \quad \Box

Proof of Lemma 5.1. Fix $p$, and let $M_{m_k}(p)$ have a convergent subsequence.
For every $m \geq 1$, we have

$$E \tilde{M}_m(p) = M_m(p)$$

$$\text{Var}(\tilde{M}_m(p)) = \frac{M_m(p)(1 - M_m(p))}{n_m}$$

For every $\epsilon > 0$, the Chebyshev inequality implies

$$\text{Prob}(|\tilde{M}_m(p) - M_m(p)| > \epsilon) \leq \frac{\text{Var}(\tilde{M}_m(p))}{\epsilon^2} = \frac{M_m(p)(1 - M_m(p))}{n_m \epsilon^2} \leq \frac{1}{4n_m \epsilon^2} \xrightarrow{m \to \infty} 0$$

Thus, $\tilde{M}_m(p) - M_m(p)$ converges to zero in probability; in particular, the subsequence $\tilde{M}_{m_k}(p) - M_{m_k}(p)$ also converges to zero in probability. The latter statement implies that the subsequence $\tilde{M}_{m_k}(p)$ converges in probability to the same limit of the converging subsequence $M_{m_k}(p)$.\[^{21}\]

Convergence in probability implies convergence in distribution. Therefore, this completes the proof. \qed

Proof of Lemma 5.2. Fix $q \in (0, Z)$, with $q \notin \mathbb{Z}^+$. For $p \notin [\text{Ask}^c, V]$, the statement is trivially true.\[^{22}\] Thus, for the remainder of this proof, $p$ is in the common support; in other words, $p \in [\text{Ask}^c, V]$. We divide the proof into the two relevant cases: $p < P^c(q)$ and $p > P^c(q)$.

Case $p < P^c(q)$

Because $\alpha(\cdot)$ is increasing and $\beta(\cdot)$ is strictly increasing, by replacing $P^c(q)$ with $p$ in the tail condition (4) we obtain

$$0 > \alpha(p) + \beta(p) \text{Prob}(q \leq \tilde{z}) \quad (18)$$

\[^{21}\]The limit of the sum equals the sum of limits. Here, the two sequences are $\tilde{M}_{m_k}(p) - M_{m_k}(p)$ and $M_{m_k}(p)$, and their sum is $\tilde{M}_{m_k}(p)$.

\[^{22}\]For $p < \text{Ask}^c \leq P^c(q)$, $\tilde{M}_m(p) \equiv 0$. Because $q > 0$, the statement is true. For $p > V > P^c(q)$, $\tilde{M}_m(p) \equiv 1$. Because $q < Z$, the statement is true.
Our goal is to show that \( \lim \text{Prob}(\tilde{M}_m(p) \leq q/Z) = 1 \); therefore, if we can show that the limit inferior of the sequence \( \text{Prob}(\tilde{M}_m(p) \leq q/Z) \) is one, then we are done. Take a subsequence that converges to the limit inferior. Any further subsequence converges to the same limit. We extract a further subsequence such that the mixing distribution at \( p \) converges. In summary, \( M_{m_k}(p) \) converges to some limit, and \( \text{Prob}(\tilde{M}_{m_k}(p) \leq q/Z) \) converges to the limit inferior of the sequence \( \text{Prob}(\tilde{M}_m(p) \leq q/Z) \). If the limit of \( M_{m_k}(p) \) is strictly smaller than \( q/Z \), then Lemma 5.1 implies \( \text{Prob}(\tilde{M}_{m_k}(p) \leq q/Z) \to 1 \), and we are done. Assume, therefore, by means of contradiction, that

\[
\lim M_{m_k}(p) \geq q/Z
\]

For every \( z \in \mathbb{Z}^+ \), with \( q > z \), we have \( \lim M_{m_k}(p) > z/Z \). Therefore, Lemma 5.1 implies that \( \text{Prob}(\tilde{M}_{m_k}(p) \leq z/Z) \to 0 \). From the Nash condition (13), we have

\[
0 = \alpha(p) + \beta(p) \text{Prob}(\tilde{M}_{m_k}(p) < \tilde{z}/Z)
= \alpha(p) + \beta(p) \sum_{z \in \mathbb{Z}^+} \text{Prob}(\tilde{z} = z) \text{Prob}(\tilde{M}_{m_k}(p) < z/Z)
\leq \alpha(p) + \beta(p) \left( \sum_{q > z, z \in \mathbb{Z}^+} \text{Prob}(\tilde{z} = z) \text{Prob}(\tilde{M}_{m_k}(p) \leq z/Z) + \sum_{q \leq z, z \in \mathbb{Z}^+} \text{Prob}(\tilde{z} = z) \right)
\downarrow \quad ||
\quad 0 \quad \text{Prob}(q \leq \tilde{z})
\rightarrow \alpha(p) + \beta(p) \text{Prob}(q \leq \tilde{z})
\]

which contradicts (18), and concludes the proof for the case \( p < P^c(q) \).

*Case \( p > P^c(q) \)^{23}

Because \( \alpha(\cdot) \) is increasing and \( \beta(\cdot) \) is strictly increasing, by replacing \( P^c(q) \) with \( p \) in the tail condition (4) we obtain

\[
0 < \alpha(p) + \beta(p) \text{Prob}(q \leq \tilde{z}) \tag{19}
\]

^{23}This is the part of the proof where we need to use the assumption that \( q \notin \mathbb{Z}^+ \).
Our goal is to show that \( \lim \text{Prob}(\tilde{M}_m(p) \leq q/Z) = 0 \). If we can show that the limit superior of the sequence \( \text{Prob}(\tilde{M}_m(p) \leq q/Z) \) is zero, then we are done. Take a subsequence that converges to its limit superior. Any further subsequence converges to the same limit. We extract a further subsequence such that the mixing distribution converges. In summary, \( M_{m_k}(p) \) converges to some limit, and \( \text{Prob}(\tilde{M}_{m_k}(p) \leq q/Z) \) converges to the limit superior of the sequence \( \text{Prob}(\tilde{M}_m(p) \leq q/Z) \). If the limit of \( M_{m_k}(p) \) is strictly greater than \( q/Z \), then Lemma 5.1 implies \( \text{Prob}(\tilde{M}_{m_k}(p) \leq q/Z) \to 0 \), and we are done. Assume, therefore, by means of contradiction, that

\[
\lim M_{m_k}(p) \leq q/Z
\]

For every \( z \in \mathcal{Z}^+ \) with \( q < z \), there is a \( q' \) such that \( q < q' < z \). Therefore, \( \text{Prob}(\tilde{M}_{m_k}(p) \leq q'/Z) \leq \text{Prob}(\tilde{M}_{m_k}(p) < z/Z) \). By contrast, \( q < q' \), so \( \lim M_{m_k}(p) < q'/Z \). Therefore, Lemma 5.1 implies that

\[
1 \leftrightarrow \text{Prob}(\tilde{M}_{m_k}(p) \leq q'/Z) \leq \text{Prob}(\tilde{M}_{m_k}(p) < z/Z) \leq 1
\]

Next, from the Nash condition (13), we have

\[
0 = \alpha(p) + \beta(p) \sum_{z \in \mathcal{Z}^+} \text{Prob}(\tilde{z} = z) \text{Prob}(\tilde{M}_{m_k}(p) < z/Z) \\
\geq \alpha(p) + \beta(p) \sum_{q < z, z \in \mathcal{Z}^+} \text{Prob}(\tilde{z} = z) \text{Prob}(\tilde{M}_{m_k}(p) < z/Z) \\
\downarrow \\
\rightarrow \alpha(p) \beta(p) \text{Prob}(q < \tilde{z}) \\
= \alpha(p) \beta(p) \text{Prob}(q \leq \tilde{z})
\]

where the last equality is true because \( q \notin \mathcal{Z}^+ \). This contradicts (19), and concludes the proof for the case \( p > P_c(q) \).

**Proof of Theorem 5.3:** Fix \( q \in (0, Z) \), with \( q \notin \mathcal{Z}^+ \). We divide the proof into the two relevant cases: \( p > P_c(q) \) and \( p < P_c(q) \).
Case $p > P^c(q)$

We need to show that $\text{Prob}(\tilde{P}^m(q) \leq p) \to 1$. We have

$$\text{Prob}(\tilde{P}^m(q) \leq p) \geq \text{Prob}(Z\tilde{M}_m(p) \geq q) \geq \text{Prob}(Z\tilde{M}_m(p) > q)$$

$$= 1 - \text{Prob}(\tilde{M}_m(p) \leq q/Z) \to 1$$

$$\downarrow$$

$$0$$

The first inequality is the left inequality of (16). The second inequality is trivial. For the limit, we use Lemma 5.2.

Case $p < P^c(q)$

We need to show that $\text{Prob}(\tilde{P}^m(q) \leq p) \to 0$. Because $q \notin \mathbb{Z}^+$, in a small neighborhood of $q$, $P^c(\cdot)$ is flat. Thus, there is a small $a > 0$ such that (i) $q - a \in (0, Z)$ with $q - a \notin \mathbb{Z}^+$, and (ii) $P^c(q) = P^c(q - a)$.

For sufficiently large $m$, we have $1/m < a$. Therefore,

$$\text{Prob}(\tilde{P}^m(q) \leq p) \leq \text{Prob}\left(Z\tilde{M}_m(p) \geq q - \frac{1}{m}\right) \leq \text{Prob}\left(Z\tilde{M}_m(p) > q - a\right)$$

$$= 1 - \text{Prob}\left(\tilde{M}_m(p) \leq \frac{q - a}{Z}\right) \to 0$$

$$\downarrow$$

$$1$$

The first inequality is the right inequality of (16). The second equality holds because we are considering only large values of $m$ such that $1/n_m < a/Z$. For the limit, we use Lemma 5.2 at price $p$ and quantity $q - a$, knowing that $p < P(q) = P^c(q - a)$. 

\[ \square \]
Maple code used in the example in Section 4

```maple
restart: with(Statistics):
V := RandomVariable(LogNormal(0, 1)):
# mu is the probability the liquidity demander is a news trader
mu := 1/3:
# PDF(V, v) is the Probability Density Function of V
alpha := p -> mu * (int((p - v) * PDF(V, v), v = p..infinity));
alpha := p -> 
mu * (int((p - v) * Statistics:-PDF(V, v) dv)
beta := p -> (1 - mu) * (p - Statistics:-Mean(V))
# competitive price of first unit
Ask := fsolve(alpha(p) + beta(p) * 1/2, p);
Ask := 2.148894561
# competitive price of second unit
fsolve(alpha(p) + beta(p) * 1/4, p)
2.506363216
# The Nash equilibrium mixing distribution
M := p -> piecewise(p < Ask, 0, 1 + alpha(p)/beta(p) * 1/2)
M := p -> piecewise(p < Ask, 0, 1 + alpha(p)/beta(p) * 1/2)
# random price distributed according to M
P := RandomVariable(Distribution(CDF = M)):
# one is the first order statistics in a sample of size 3, two and three are the second and third
Mean(one), Mean(two), Mean(three)
2.320657094, 2.670879942, 3.837032686
# plot(PDF(one, p), PDF(two, p), PDF(three, p), p = Ask..5); # Compute the 99% confidence interval for the best asking price
fsolve(CDF(one, p) = 0.005), fsolve(CDF(one, p) = 1 - 0.005);
2.149578757, 3.411256976
# Compute the 99% confidence interval for the second asking price
fsolve(CDF(two, p) = 0.005), fsolve(CDF(two, p) = 1 - 0.005);
2.166449935, 5.434745336
fsolve(CDF(three, p) = 0.005), fsolve(CDF(three, p) = 1 - 0.005);
2.230692242, 14.84086766
```

(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
(9)
References


