The Limit Order Book and the Break-Even Conditions, Revisited

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We model a limit order market with strategic liquidity suppliers. Using the Bellman equation, we identify a candidate to be the equilibrium limit order book. As the number of liquidity suppliers increases, the candidate converges to a limit order book that satisfies the break-even conditions (i.e., the competitive equilibrium). However, we show that in some relevant economic environments the Bellman equation characterizes an argument of minimum rather than maximum of the liquidity suppliers' objective function. Thus, the intuition that underlies the break-even conditions is incomplete. In particular, the results in this paper are in conflict with the results in Biais, Martimort, and Rochet (2000).

The assumption that liquidity suppliers break even provides a unifying framework for the literature on market microstructure. The break-even conditions state that prices equal the expected value of the asset conditional on information contained in the order flow.

In a market organized as a pure limit order book, a market order walks the book, picking off limit orders at their limit-prices. In that context, the liquidity suppliers are the limit order submitters. Following Rock (1990) and Glosten (1994), it is common to express the break-even conditions in terms of upper and lower tail expectations of the asset value, where the conditioning is on execution.

[1] I thank Kerry Back, Dan Bernhardt, Mark Loewenstein, Gideon Saar, and seminar participants at the University of Utah, Cornell University, and BYU.

The goal of this paper is to reassess the Rock-Glosten break-even conditions. To that end, we study a model of imperfect competition in liquidity provision and ask whether the competitive equilibrium is the limit of the model with a large number of liquidity suppliers.

As in Bernhardt and Hughson (1997) and Biais, Martimort, and Rochet (2000) (hereafter BMR), we model the interaction between the liquidity suppliers as a static game, in which the liquidity suppliers compete in supply functions. After supply functions are submitted, an active trader submits a marketable order. The active trader may have private information about the value of the asset, but may also trade for liquidity reasons.

We use the principle of optimality to examine the problem of a liquidity supplier.[1] The Bellman equation can be used to construct a candidate for equilibrium, and we see that the aggregate supply function satisfies, at the limit when the number of liquidity suppliers is large, the Rock-Glosten conditions. However, it is not clear that the candidate is indeed an equilibrium.

We find that the problem of a liquidity supplier is locally linear, which implies that the Bellman equation is degenerate. That is, a priori we don't know whether the candidate for equilibrium is an outcome of a maximization or a minimization effort.[2] We apply a two dimensional version of the fundamental theorem of calculus (Green's

[1]The problem is not dynamic in time. We just exploit the structure of the problem and use the cumulative depth as the state variable.

[2]Loosely speaking, if the candidate is an argument of minimum rather than maximum, then at the limit, when the number of liquidity suppliers is large, all "losses" (rather than gains) are competed away.

theorem) to state sufficient conditions for equilibrium.[3] We then compute the equilibrium in several examples. However, we find that the sufficient conditions are not merely technical. In fact, in some relevant economic environments the candidate we derived using the Bellman equation is an argument of minimum, for example, when the liquidity suppliers face the risk of trading with news traders (as in Copeland and Galai (1983) or the uniform example in Glosten (1994)).

Is it still possible, in those economic environments where the Bellman equation fails to characterize the equilibrium, that the Rock-Glosten break-even conditions are satisfied at the limit of equilibria that we don't know how to compute? To answer this question, we prove a non-existence result that is related to the competition for precedence among the liquidity suppliers.[4] In economic environments in which precedence is even more important further back in the book, no equilibrium with piecewise differentiable supply functions exists. Intuitively, the equilibrium fails to exist in those economic environments where the ratio of informed to uninformed orders, conditional on the size of the order, is increasing with the size of the order.[5]

A survey of the literature on the limit order book is given in Parlour and Seppi (2008). This paper adds in particular to the literature on the limit order book and adverse selection. Models that postulate

[3]The fundamental theorem of calculus states that a function is the integral of its derivative. The use of Green's theorem to solve locally linear problems is due to Miele (1962).

[4]The precedence of an offer is defined to be the number of shares offered at better prices.

[5]Glosten (1989) discusses a market breakdown condition due to severe adverse selection. Here, however, the competitive equilibrium exists.

the break-even conditions include Copeland and Galai (1983), Rock (1990), Glosten (1994)), and the dynamic model of Back and Baruch (2007). Sandas (2001) empirically tests and rejects the restrictions implied by the break-even conditions.

Bernhardt and Hughson (1997) show that the competitive book is not the outcome of a model with a finite number of strategic liquidity suppliers. BMR study one economic environment (the exponential environment) and conclude that the competitive equilibrium is the limit of a model with large number of liquidity suppliers. However, the technical conditions BMR impose are not equivalent to the conditions we get from Green's theorem. We show (in Appendix C) that the Bellman equation characterizes the "equilibrium" BMR find, and we give an example that satisfies all the technical conditions they impose though a profitable deviation exists.

The paper is organized as follows. In Section 1 we set the model. In section 2 we describe different economic environments. In Section 3 we state the Rock-Glosten break-even conditions. In Section 4 we use the Bellman equation to derive a candidate for equilibrium. In Section 5, using Green's theorem, we provide sufficient conditions for equilibrium. In Section 6 we prove a non-existance result. In Section 7 we conclude.

1. THE MODEL

We model a pure limit order market for a single risky asset. There are two types of traders in the model: n risk neutral, uninformed liquidity suppliers and a single active trader. Initially, the liquidity suppliers submit limit orders, and then the active trader, after observing all bids and offers in the book, submits a marketable order. Following the trade, the asset is liquidated; the liquidation value is $\widetilde{v}$. Without loss of generality, we focus on the offer side of the book.[6]

We follow the modeling choice of Bernhardt and Hughson (1997) and BMR, and model the interaction between the liquidity suppliers as a static game in which offers are posted simultaneously. We assume that there is a maximum price, which we denote by p_{max}, above which offers cannot be posted.[7] Because each liquidity supplier can post multiple offers, a strategy for the i-th liquidity supplier is a supply function $S_i : [0, p_{max}] \to R$ with the interpretation that $S_i(p)$ is the cumulative number of shares offered up to the price p. Equivalently, "$dS_i(p)$" is the number of shares offered in the price interval "$(p, p + dp)$." We say that a supply function is *feasible* if it is non-decreasing and continuous.[8] For a given profile of strategies $(S_1, S_2, \dots, S_n)$, we let $S_{-i} = \sum_{j \neq i} S_j$.

We assume that there is a function $u : R_+^2 \to R$ such that, for every i, the ex-ante expected payoff of the i-th liquidity supplier is given by

$$\int_0^{p_{max}} u(p, S(p)) dS_i(p) \tag{1.1}$$

[6]We don't lose generality because the liquidity suppliers are uninformed.

[7]The assumption that there is a maximum price allows us to postpone the discussion about the endogenous maximum price, while at the same time allows us to solve the model when, for technical reason, a maximum price is imposed.

[8]Though we restrict the class of feasible supply functions to those that are continuous, we don't impose any upper bound on "dS". Therefore, in the equilibrium we find, it is not optimal to post discrete offers.

where

$$S(p) = S_i(p) + S_{-i}(p) \tag{1.2}$$

We call S the *aggregate supply function*, and we call u the *profitability function*. In the next section we provide examples of plausible economic environments and the profitability functions associated with them.

A *competitive equilibrium* is an aggregate supply function S and a profile of feasible strategies, $(S_1, S_2, \ldots, S_n)$, such that (i) for each i, S_i maximizes the payoff (1.1), taking the aggregate supply function S as given, and (ii) (1.2) is satisfied.

A *Nash equilibrium* is a profile of feasible strategies, $(S_1, S_2, \ldots, S_n)$, such that for each i, S_i maximizes the payoff (1.1) subject to the constraint (1.2).

A Nash equilibrium, $(S_1, S_2, \ldots, S_n)$, is *piecewise differentiable* if for each i there exists a piecewise continuous function s_i such that $S_i(p) = \int_0^p s_i(x)dx$.

A *symmetric equilibrium* is an equilibrium in which all the liquidity suppliers use the same strategy.

2. THE ECONOMIC ENVIRONMENT

The profitability function, u, depends on the economic environment which we have not yet fully specified. Relevant factors that determine u are the distribution of the liquidation value, the information available to the active trader, the active trader's attitude toward risk, etc. To the best of my knowledge, all the examples in the literature are special cases of the environments we describe below.

2.1. *News Traders*

This environment was first considered in Copeland and Galai (1983). Here trade takes place because either a liquidity event or an informational event occurs. Conditional on occurrence, with probability μ it is a liquidity event and with probability $1-\mu$ it is an informational event.

A liquidity event occurs when an uninformed trader desires to trade for individual motives. We assume that the uninformed trader submits a limit order with a size of $\widetilde{q}$ shares and a limit price $\widetilde{p}$. If the order is a buy order and there are offers in the book at prices lower than $\widetilde{p}$, then the limit order is marketable and the order walks up the book until it is completely filled or it reaches its limit price, whichever comes first.

An informational event occurs when news, relevant for the value of the asset, is not disseminated simultaneously.[9] We assume that, perhaps just an instant before the liquidity suppliers learn the news and have the chance to refresh their quotes, news traders pick off all stale bids and offers. For simplicity, we assume the liquidation value, $\widetilde{v}$, is revealed to the news traders.

[9]For example, Jack Dorsey, the CEO of Twitter, said in an interview that "The minute the Los Angeles earthquake struck [in July 2008] there was an update on Twitter, which was followed by thousands of more updates, until nine minutes later the first reports came out on the AP wire" (see Phillips (2008)).

Because offers are filled at their offering prices, the profitability function associated with the offer side of the book is

$$\begin{aligned} u(p,q) &= E\left[(1-\mu)(p-\widetilde{v})I_{\{\widetilde{v}>p\}} + \mu(p-\widetilde{v})I_{\{\widetilde{p}>p,\widetilde{q}>q\}}\right] \\ &= (1-\mu)E(p-\widetilde{v})I_{\{\widetilde{v}>p\}} + \mu(p-E\widetilde{v})EI_{\{\widetilde{p}>p,\widetilde{q}>q\}} \end{aligned}$$

In the *uniform example* in Glosten (1994), the liquidation value is uniformly distributed over $[-L,L]$, and the joint distribution of $\widetilde{p}$ and $\widetilde{q}$ satisfies [10]

$$EI_{\{\widetilde{p}>p,\widetilde{q}>q\}} = \begin{cases} \frac{L-(p+q)}{2L} & p+q \leq L \\ 0 & p+q > L \end{cases}$$

Hence, in Glosten's uniform example, the profitability function is

$$u(p,q) = \begin{cases} -\frac{1-\mu}{4L}\left(L-p\right)^2 + \frac{\mu}{2L}p(L-(p+q)) & p+q \leq L \\ -\frac{1-\mu}{4L}\left(L-p\right)^2 & p \leq L \\ 0 & \text{, otherwise} \end{cases} \tag{2.1}$$

2.2. *Exponential*

This economic environment was considered in Glosten (1994) and was at the center of BMR's paper. The active trader has an exponential utility with a coefficient of risk aversion γ and an initial position of $\widetilde{I}$ shares in the asset. The liquidation value of the asset

[10] In other words, an uninformed trader's order walks up the book as long as $\widetilde{\epsilon} > p + S(p)$ where $\widetilde{\epsilon}$ is uniformly distributed $[-L,L]$. In Glosten's terminology, the marginal valuation of an uninformed trader is $\epsilon - S(p)$.

is $\widetilde{v} = \widetilde{v}_s + \widetilde{\epsilon}$, and the active trader observes the signal $\widetilde{v}_s$. The noise term, $\widetilde{\epsilon}$, is a zero mean normal random variable with variance σ^2, and $\widetilde{\epsilon}$ is independent of $\widetilde{v}_s$ and $\widetilde{I}$. The signal, $\widetilde{v}_s$, and the initial position, $\widetilde{I}$, can be dependent. Define the summery statistics

$$\widetilde{\theta} \equiv \widetilde{v}_s - \gamma\sigma^2 \widetilde{I}$$

LEMMA 1 *Given any non-decreasing aggregate supply function, S, the active trader's order walks up the book as long as $\widetilde{\theta} > \gamma\sigma^2 S(p) + p$.*

The proof of the lemma is in Appendix A.

The random variable $\widetilde{\theta}$ is a summary statistics in the sense that there is no more information in the order flow about the liquidation value beyond the information in $\widetilde{\theta}$. Let F be the distribution function of $\widetilde{\theta}$, and define the functions $v(\theta) = E[\widetilde{v}|\widetilde{\theta} = \theta]$ and $v^+(\theta) = E[\widetilde{v}|\widetilde{\theta} > \theta]$. The profitability function is then

$$\begin{aligned} u(p,q) &= E(p - \widetilde{v}) I_{\{\widetilde{\theta} > \gamma\sigma^2 q + p\}} \\ &= p\left(1 - F\left(\gamma\sigma^2 q + p\right)\right) - \int_{\gamma\sigma^2 q + p}^{\infty} v(\theta) dF(\theta) \\ &= \begin{cases} \left(p - v^+\left(\gamma\sigma^2 q + p\right)\right)\left(1 - F\left(\gamma\sigma^2 q + p\right)\right) & , \gamma\sigma^2 q + p \leq \bar{\theta} \\ 0 & , \text{otherwise} \end{cases} \end{aligned} \tag{2.2}$$

where $\bar{\theta}$ is the upper support of $\widetilde{\theta}$. In the *exponential-normal* example in Glosten (1994), $\widetilde{v}_s$ and $\widetilde{I}$ are assumed to be independent and normally distributed. In BMR, the support of $\widetilde{v}_s$ and $\widetilde{I}$ are closed intervals.

2.3. *Inelastic Demand*

In this economic environment, the size of the active trader's order, $\widetilde{q}$, is independent of the posted bids and offers. Let F be the cumulative distribution function of the random order size $\widetilde{q}$, let $v(q) = E[\tilde{v}|\widetilde{q} = q]$, and let $v^+(q) = E[\tilde{v}|\widetilde{q} > q]$.[11] The profitability function is

$$\begin{aligned} u(p,q) &= E(p - \widetilde{v})I_{\{\widetilde{q}>q\}} = p(1 - F(q)) - \int_q^\infty v(u)dF(u) \\ &= (p - v^+(q))(1 - F(q)) \end{aligned}$$

3. THE COMPETITIVE EQUILIBRIUM

Given the profitability function, u, it is easy to characterize the competitive equilibrium. A non-decreasing function, S, is the aggregate supply function in the competitive equilibrium if and only if the break-even conditions hold; i.e.

$$u(p, S(p)) \leq 0$$

with equality whenever $dS(p) > 0$. We may have a strict inequality because there are prices at which even competitive liquidity suppliers do not offer shares (e.g., prices that are below the ask price).

The competitive equilibrium may not make economic sense unless we impose additional conditions on the economic environments such as the "strict adverse selection" condition in Glosten (1994). This

[11] We use the same notation as in the exponential environment, but there will be no ambiguity because we study the environments separately.

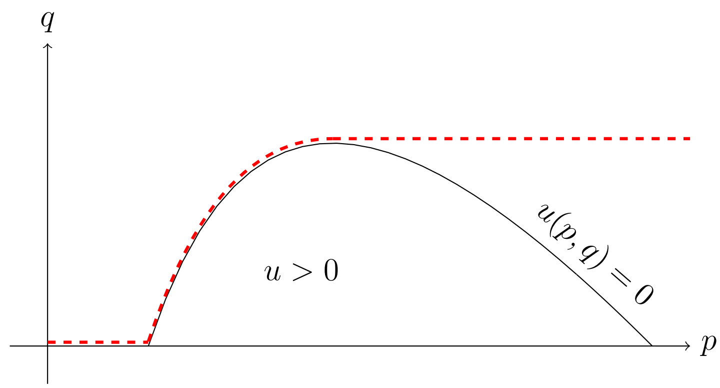

FIGURE 1.— The Region $\{u > 0\}$ in the Uniform Example. The dotted graph is the graph of the feasible supply function that satisfies the break-even condition. The profitability function is given in (2.1), and the parameters are $L = 1$ and $\mu = 1/2$.

specific condition is expressed in terms of Glosten's marginal valuation function, which we have not defined in this paper.[12] Instead, we say that an economic environment is *regular* if $\{u > 0\} \equiv \{(p, q) : u(p, q) > 0\}$ is the region bounded from above by the curve $u(p, q) = 0$ and from below by the p-axis. In a regular economic environment, we can think about competitive liquidity suppliers submitting offers until all profits are competed a way. Though we are not explicitly assuming that the economic environment is regular, all the examples we consider satisfy this condition. In Figure (1), we show the region $\{u > 0\}$ and the feasible supply function that satisfies the break-even conditions in Glosten's uniform example.

[12]Whereas in Glosten (1994) the liquidity suppliers take the active's trader marginal valuation as given, in this paper the liquidity suppliers take the profitability function as given. The advantage of our approach is that we can now also study the inelastic demand environment, which is an environment where marginal valuations are not defined.

4. THE BELLMAN EQUATION

Let p_{ask} denote the lowest price at which offers are posted; i.e., the ask price. In this section, we use the Bellman equation to informally derive a candidate for a symmetric Nash equilibrium in which the aggregate supply function is strictly increasing in the interval of prices $[p_{ask}, p_{max}]$.

Even though the problem of a liquidity supplier is static, we can nevertheless use the principle of optimality, as if a liquidity supplier first chooses how many shares to offer at lower prices. Write $S_i(p) = \int_0^p s_i(x)dx$ and define[13]

$$V(p_0, q_0) = \begin{cases} \max\limits_{s_i(p)\geq 0} \int_{p_0}^{p_{max}} u(p, S(p))s_i(p)dp \\ \text{subject to} \\ dS = s_{-i}(p) + s_i(p)dp \\ S(p_0) = q_0 \end{cases}$$

The Bellman equation is

$$\max_{s_i\geq 0} V_p(p, q) + V_q(p, q)(s_{-i}(p) + s_i) + u(p, q)s_i = 0$$

or, after arranging terms,

$$V_p(p, q) + V_q(p, q)s_{-i}(p) + \max_{s_i\geq 0} \left(V_q(p, q) + u(p, q)\right) s_i = 0$$

[13] A feasible supply function is non-decreasing and continuous and hence there exists a function s_i such that $S_i(p) = \int_0^p s_i(x)dx$.

Because the problem is linear in s_i, in the interval of prices $[p_{ask}, p_{max}]$, we must have

$$\begin{aligned} V_q(p,q) =& - u(p,q) \\ V_p(p,q) =& - V_q(p,q)s_{-i}(p) \end{aligned}$$

In the second equality, we substitute for V_q the value $-u$ (from the first equality) and get an equivalent system:

$$\begin{aligned} V_q(p,q) =& - u(p,q) \\ V_p(p,q) =& s_{-i}(p)u(p,q) \end{aligned}$$

Now, differentiate the first equation by p and the second by q, to get

$$\begin{aligned} V_{qp}(p,q) =& - u_p(p,q) \\ V_{pq}(p,q) =& s_{-i}(p)u_q(p,q) \end{aligned}$$

So, for the i-th liquidity supplier to have an optimal strictly positive solution, the equilibrium supply function must solve the equation

$$s_{-i}(p) = -\frac{u_p(p, S(p))}{u_q(p, S(p))}$$

The right hand side is independent of i, which is consistent with the symmetry of the equilibrium. Thus, we set $s_{-i}(p) = \dot{S}(p)(n-1)/n$.

We conclude that the equilibrium aggregate supply function solves the free boundary problem

$$\begin{aligned} \dot{S}(p) &= -\frac{n}{n-1}\frac{u_p(p,S(p))}{u_q(p,S(p))}, && p \in (p_{ask}, p_{max}) \\ S(p) &= 0, && p \in (0, p_{ask}) \end{aligned} \tag{4.1}$$

subject to the boundary condition $S(p_{max}) = q_{max}$, where

$$q_{max} \equiv \underset{q \geq 0}{\text{argmax}} \int^{q} u(p_{max}, y)dy \tag{4.2}$$

To understand the boundary condition intuitively, we note that the solution to (4.1) is a viable candidate to be the equilibrium aggregate supply function only if S is strictly increasing in $[p_{ask}, p_{max}]$. In particular, $S(p_{max}) = q_{max} > 0$. But if $q_{max} > 0$, then the definition of $q_{\max}$, (4.2), implies $u(p_{max}, q_{max}) = 0$. In regular economic environments, to offer less than q_{max} shares would mean to leave money on the table, while to offer more than q_{max} would imply losses. Thus we expect that exactly q_{max} shares will be offered. We will verify the boundary condition in Theorem 5.1 in the next section.

The differential equation (4.1) is a free boundary problem because as part of the solution also the ask price, p_{ask}, has to be determined.

As we let the number of liquidity suppliers, n, go to infinity, the problem (4.1) becomes

$$\begin{aligned} \dot{S}(p) &= -\frac{u_p(p,S(p))}{u_q(p,S(p))}, && p \in (p_{ask}, p_{max}) \\ S(p) &= 0, && p \in (0, p_{ask}) \end{aligned}$$

subject to the boundary condition $S(p_{max}) = q_{max}$. In the interval

(p_{ask}, p_{max}), the solution is given implicitly by the equation $u(p, S(p)) = 0$. Intuitively, this means that the sequence of candidates for equilibrium we have identified converges to the competitive equilibrium as the number of liquidity supplier increases.[14]

The Uniform Example:

To demonstrate the convergence of the candidate to the competitive equilibrium, we consider Glosten's uniform Example. The profitability function, u, is given in (2.1). We set $L = 1$ and $\mu = 1/2$. The aggregate supply function that satisfies the break-even conditions is given by (see also Figure 1)

$$S_\infty(p) = \begin{cases} 0 & p < 1/3 \\ 2 - \frac{3}{2}p - \frac{1}{2p} & 1/3 \leq p \leq \frac{1}{\sqrt{3}} \\ 2 - \sqrt{3} & \frac{1}{\sqrt{3}} \leq p \end{cases}$$

In particular, shares are not offered at prices greater than $1/\sqrt{3}$. This book is what Glosten (1994) reports in page 1144.[15]

We set the exogenous maximum price to $p_{max} = 1/\sqrt{3}$ and find that $q_{max} = 2 - \sqrt{3}$. To solve the free boundary problem (4.1), we solve the o.d.e. backward until the solution vanishes. The solution is given

[14] To make this statement rigorous, one has to impose additional technical assumptions on the profitability function u.

[15] In our analysis, the marginal price p is the state variable, while in Glosten (1994) the total number of shares, denoted there by δ, is the state variable. Replace δ with $S(p)$ and p with $R'(\delta)$ to see that the competitive books are identical.

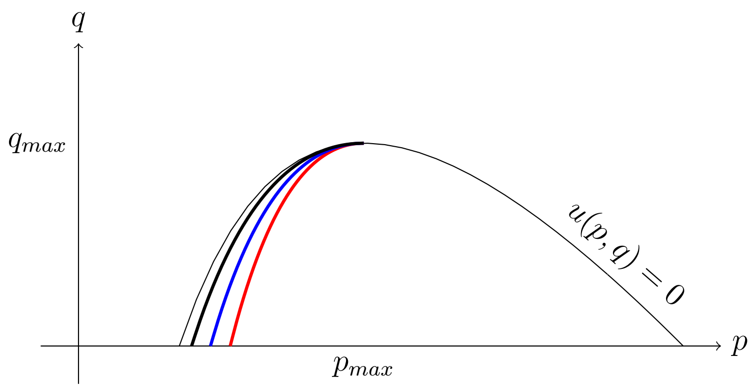

FIGURE 2.— A Solution of (4.1) in the Uniform Example. The lower curve corresponds to $n = 2$, and then, in ascending order, $n = 3$, $n = 7$, and the curve $u(p,q) = 0$. The figure illustrates the convergence of the candidates to the competitive equilibrium. Later we show that those candidates are not equilibrium aggregate supply functions. As in Figure (1), the parameters are $\mu = 0.5$ and $L = 1$. The exogenous maximum price at which liquidity suppliers can post offers is $p_{max} = 1/\sqrt{3}$, which is also the endogenous maximum price at which offers are posted in the competitive equilibrium.

by

$$S^*(p) = \max\left\{0, 2 - \frac{3np}{2n-1} - \frac{(n-1)}{(2n-1)} p^{-n/(n-1)} 3^{-1/(2n-2)}\right\}$$

We cannot solve for the ask price analytically, but we can easily do it numerically. For example, for $n = 2$, the ask price is $p^*_{ask} = 0.4007035757$.

The solution, S^*, is strictly increasing above the ask price, and as we increase n, the solution converges to the one that satisfies the break-even conditions (see Figure (2)). However, in Section 6, we will show that S^* is not the equilibrium aggregate supply function. In fact, in the uniform example, the Bellman equation identifies an argument of minimum rather than the argument of maximum of the liquidity suppliers' objective function. Indeed, in the analysis of the Bellman equation we carried above, nowhere have we used the fact that the objective is to maximize gains, rather than say minimize them. Moreover, because the Bellman equation is linear, there are no second order conditions that can be checked. The standard approach in dynamic programming is to use the value function to verify the solution. Here, however, the value function is only identified along the "equilibrium path." That is, we can calculate $V(p, S(p))$, but we don't know what $V(p, q)$, for arbitrary pairs of (p, q), is. We therefore have to use a different approch to verify the equilibrium. We do so in the next section.

5. A VERIFICATION THEOREM

In this section we provide sufficient conditions for equilibrium. Since an equilibrium strategy is a global optimum of a liquidity supplier's objective function, we need to examine the partial derivatives of the profitability function everywhere. However, in many relevant economic environments the profitability function u is not differentiable everywhere.[16] That said, the method of proof we will be using is a two-dimensional version of the fundamental theorem of calculus (i.e. a function is an integral of its derivative), and to invoke the fundamental theorem, all we need is absolute continuity (AC). Absolute

[16]E.g., when the random variables that underly the economic environment are finite.

continuity is a condition stronger than continuity but weaker than differentiability.

(AC): *There are two functions f and g such that*

$$u(p,q) = u(0,q) + \int_0^p f(x,q)dx$$

and also

$$u(p,q) = u(p,0) + \int_0^q g(p,y)dy$$

The (AC) assumption implies that for every q, $p \to u(p,q)$ is absolutely continuous, and for every p, $q \to u(p,q)$ is absolutely continuous. In particular, u is continuous, and the partial derivatives exist almost everywhere.

We denote by S^* and p^*_{ask} the solution to the free boundary problem (4.1). Let

$$s^*(p) = \begin{cases} 0, & p < p^*_{ask} \\ -\frac{n}{n-1}\frac{u_p(p,S^*(p))}{u_q(p,S^*(p))}, & p^*_{ask} \le p \le p_{max} \end{cases} \tag{5.1}$$

so that $S^*(p) = \int_0^p s^*(x)dx$. Next, let Miele's function be given by

$$\omega^*(p,q) \equiv u_p(p,q) + u_q(p,q)\frac{(n-1)}{n}s^*(p)$$

In particular, we have

(5.2)

$$\omega^*(p,q) = u_p(p,q) - u_q(p,q)\frac{u_p(p,S^*(p))}{u_q(p,S^*(p))}, \qquad p \in (p^*_{ask}, p_{max})$$

We state the following condition on the sign (SGN) of Miele's function.

(SGN): *The sign of Miele's function is negative below the graph of* S^* *and positive above it.*

THEOREM 5.1 *Let* S^* *be a solution of* (4.1). *If (i) the profitability function,* u*, satisfies the (AC) condition, (ii)* S^* *is a non-decreasing function, and (iii) Miele's function satisfies the (SGN) condition, then* S^*/n *forms a symmetric Nash equilibrium and* S^* *is the equilibrium aggregate supply function.*

The first condition is technical. We need an assumption stronger than continuity to carry out the analysis in the proof. The second condition ensures that the individuals' supply functions are feasible. The third condition is perhaps unusual but has a simple interpretation. Consider an arbitrary function, f, that is absolutely continuous on the interval $[a, b]$. If we were to plot the sign of f', and find an $x \in [a, b]$ such that in $[a, x)$, the sign of f' is positive and in $(x, b]$ the sign is negative, then x maximizes f in the interval. Even if x is at the corner of the interval or f' does not exist at x (i.e., the local conditions for optimality fail), x is optimal. This result is a consequence of the fundamental theorem of calculus: for any $z \in [a, b]$, $f(x) - f(z) = \int_z^x f'(u)du > 0$.

In the proof of Theorem 5.1, we express the objective of a liquidity supplier as a line integral along a curve, and the goal is to choose the curve that maximizes the line integral. We will see that Miele's function is analogous to the derivative of the line integral, and therefore we are interested in the sign of Miele's function.

Before we prove the theorem, we show how we apply the theorem in two examples.

The Exponential-Normal Example:

Glosten (1994) studies the competitive equilibrium in a special case of the exponential environment. In that case, $\widetilde{I}$ and $\widetilde{v}_s$ are normally distributed with zero mean and covariance. The variance of $\widetilde{I}$ is $\alpha < 1$, the variance of $\widetilde{v}_s$ is $1-\alpha$, and $\sigma^2\gamma = 1$, so that $\widetilde{\theta}$ is a standard normal random variable.[17] Thanks to the normality of the random variables, we have

$$v(\theta) = E[\widetilde{v}_s|\widetilde{\theta} = \theta] = E\widetilde{v}_s + \frac{Cov(\widetilde{v}_s, \widetilde{\theta})}{Var(\widetilde{\theta})}\left(\theta - E\widetilde{\theta}\right) = (1-\alpha)\theta$$

and

$$v^+(\theta) = \frac{\int_\theta^\infty v(t)\varphi(t)dt}{1-\Phi(\theta)} = (1-\alpha)\frac{\varphi(\theta)}{1-\Phi(\theta)}$$

[17] To see that this is indeed what Glosten (1994) assumes, we note the following. In Glosten's original description of the example, the active trader observes $\widetilde{\text{Signal}} = \widetilde{v} + \widetilde{\text{noise}}$, where the liquidation value and the noise are independent and normally distributed with zero mean. Moreover, the variance of $E[\widetilde{v}|\widetilde{\text{Signal}}]$ is $1-\alpha$, and the standard deviation of $\widetilde{v}$, conditional on the signal, is σ. By the law of total variance, it follows that the liquidation value has a variance of $1-\alpha+\sigma^2$.

To see the equivalence between our setting and Glosten's, we decompose the liquidation value into a sum of two independent normal random variables:

$$\widetilde{v} = E[\widetilde{v}|\widetilde{\text{Signal}}] + (\widetilde{v} - E[\widetilde{v}|\widetilde{\text{Signal}})$$

and call the first term $\widetilde{v}_s$ (i.e. $\widetilde{v}_s = E[\widetilde{v}|\widetilde{\text{Signal}}]$) and the second one $\widetilde{\epsilon}$ (i.e. $\widetilde{\epsilon} = \widetilde{v} - E[\widetilde{v}|\widetilde{\text{Signal}}]$). The random variables $\widetilde{v}_s$ and $\widetilde{\text{Signal}}$ are informationally equivalent, thus we can assume the active trader observes $\widetilde{v}_s$ rather than the original signal. The variance of $\widetilde{v}_s$ is $1-\alpha$ by definition and because $\widetilde{\epsilon}$ and $\widetilde{v}_s$ are independent, the variance of $\widetilde{\epsilon}$ is σ^2. Thus the setting we describe is indeed Glosten's exponential-normal example.

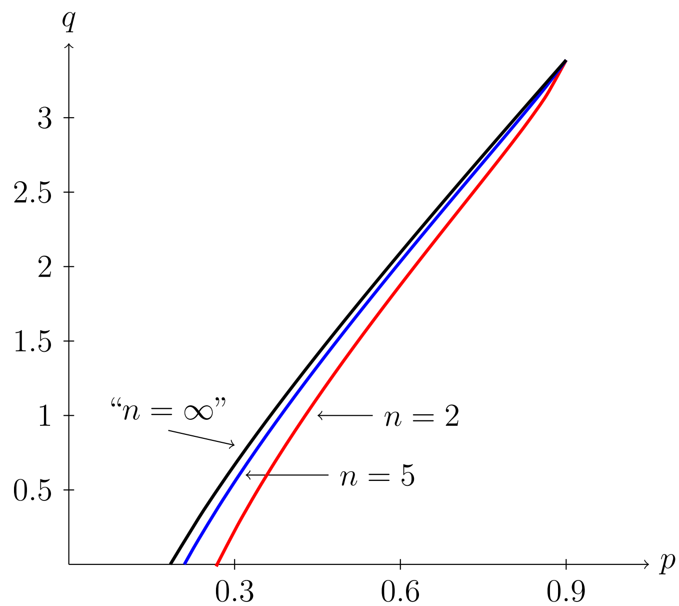

FIGURE 3.— Equilibrium Aggregate Supply function in the Exponential-Normal Example. This figure shows the equilibrium aggregate supply function for different numbers of liquidity suppliers. The parameters are $\alpha = 0.8$ and $p_{max} = 0.9$, and n is either $2, 5$ or the limiting case that satisfies the break-even conditions ("$n = \infty$").

where φ and Φ are the density function and the cumulative distribution function of a standard normal random variable, respectively. Thus, the profitability function u is given by

$$u(p, q) = p(1 - \Phi(q + p)) - (1 - \alpha)\varphi(q + p).$$

In particular, u is differentiable everywhere and hence u satisfies (AC).

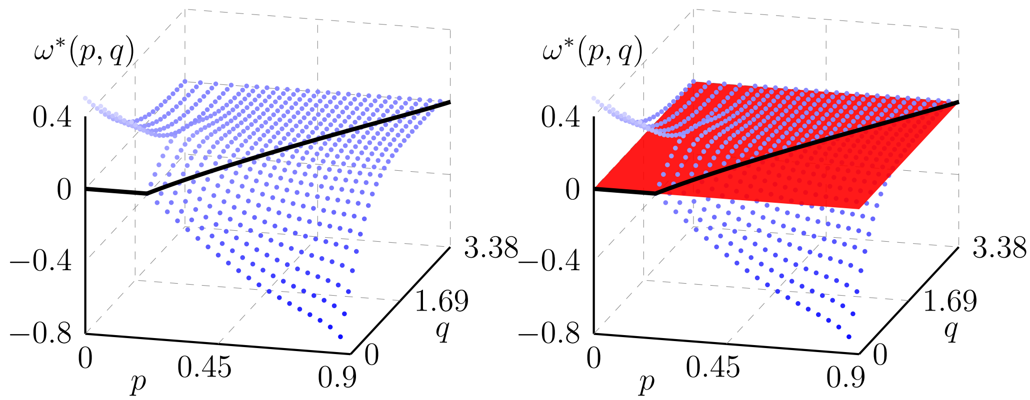

FIGURE 4.— Miele's Function in the Exponential-Normal Example. This figure demonstrates that the (SGN) holds in the exponential-normal example. The equilibrium aggregate supply function, S^*, is depicted in the plane $(p, q, 0)$. For better visualization, in the right panel we also show the surface $(p, q, 0)$. It can be seen that the sign of ω^* is positive at (p, q) if and only if (p, q) is above (in the pq-plane) the graph of S^* (i.e. iff $q > S^*(p)$). The parameters are $\alpha = 0.8$ and $p_{max} = 0.9$. The number of liquidity suppliers is $n = 5$.

We set the parameters: $\alpha = 0.8$ and $p_{max} = 0.9$. Because $q \to u(p_{max}, q)$ changes signs only once, q_{max} is the root of the equation $u(0.9, q) = 0$. We solve and find that $q_{max} = 3.387092$. Next we solve (4.1) by integrating backward the o.d.e. until the solution vanishes. Figure (3) shows the solution for different number of value traders. In particular, the solution is non-decreasing and hence it is a feasible supply function. Thus, to verify that we have found the equilibrium, we only need to check that the sign of ω^* is positive above the graph of S^* and negative below it. Figure (4) shows that this is indeed the case.

An Inelastic Demand Example:

Assume the size of the active trader's orders is uniformly distributed over the unit interval, and $v(q) = q$. We set the maximum price $p_{max} = 1$. The profitability function is therefore given by

$$u(p, q) = \begin{cases} p(1-q) - 1/2 + q^2/2 & q \leq 1 \\ 0 & q > 1 \end{cases}$$

To see that u satisfies (AC), we note that for all p,

$$u(p, q) = p - 1/2 + \int_0^q I_{\{q \leq 1\}}(y - p)dy$$

and hence $q \to u(p, q)$ is absolutely continuous. Also, for every q, $p \to u(p, q)$ is continuously differentiable and hence $p \to u(p, q)$ is absolutely continuous. We conclude that u satisfies the (AC) condition.

To find q_{max}, we solve $u(p_{max}, q) = 0$ and find that $q_{max} = 1$. The solution of the backward differential equation in (4.1) is linear:

$$S^*(p) = p\frac{2n-1}{n-1} - \frac{n}{n-1}$$

and the ask price is the value at which the solution vanishes:

$$p^*_{ask} = \frac{n}{2n-1}$$

Note that

$$\lim_{n \to \infty} S^*(p) = \begin{cases} 2p - 1 & 1/2 \leq p \leq p_{\max} \\ 0 & 0 \leq p \leq 1/2 \end{cases}$$

which is a supply function that satisfies the break-even conditions.

To verify that S^* is the equilibrium aggregate supply function, note that for $q > 1$, $\omega^* \equiv 0$. For $q \leq 1$, Miele's function is

$$\omega^*(p,q) = \begin{cases} 1-q, & 0 \leq p < p^*_{ask} \\ (1-q) - (q-p)\frac{(1-2n)}{n}, & p^*_{ask} \leq p \leq 1 \end{cases}$$
$$= \begin{cases} 1-q, & 0 \leq p < p^*_{ask} \\ \frac{n}{n-1}\left(q - S^*(p)\right), & p^*_{ask} \leq p \leq 1 \end{cases}$$

Because $S^*(p) = 0$ for every price p below the ask price, we conclude that ω^* is positive above the graph of S^* and negative below it. Thus, we have verified the equilibrium in this example.

Proof of Theorem 5.1: Assume the conditions in the theorem hold, and $n-1$ liquidity suppliers use the strategy S^*/n. We need to show that S^*/n is optimal for the i-th liquidity supplier.

Let S_i be an arbitrary feasible supply function, and let $S = S_i + \frac{n-1}{n}S^*$ be the aggregate supply function associated with S_i. Recall that s^* was defined to be such that $S^*(p) = \int_0^p s^*(x)dx$ (see (5.1)). So by inserting the constraint (1.2) into the objective (1.1), we remove the individual supply function, S_i, from the objective and express the payoff of the i-th liquidity supplier in terms of the aggregate supply function:

$$I(S) \equiv \int_0^{p_{max}} u(p,S(p))dS(p) - \int_0^{p_{max}} u(p,S(p))\frac{(n-1)}{n}s^*(p)dp$$

Our goal is to show that $I(S^*) \geq I(S)$ for every feasible supply function S.[18]

Let

$$Q(p,q) = u(p,q)$$
$$P(p,q) = -u(p,q)\frac{(n-1)}{n}s^*(p)$$

and note that $I(S)$ is the line integral of $Qdq + Pdp$ along the graph of the supply function $S(p)$.[19]

Consider now a feasible supply function, S, with a graph that lies above the graph of S^*, as shown in Figure (5). Let α consists of all those points that are above the graph of S^* and below the graph of S; i.e.

$$\alpha = \{(p,q) : p_0 \le p \le p_{max}, S^*(p) \le q \le S(p)\}$$

Let $\partial\alpha$ be the counterclockwise oriented curve that forms the boundary of α, and define the vertical plane curve $C = \{(p_{max}, q) : q_{max} \le q \le S(p_{max})\}$.

[18]This problem is more general than the original problem because we don't require that S_i is non-decreasing; i.e. we don't require that $dS(p) \ge dS^*(p)(n-1)/n$. We ignore this condition because it was never binding in specific examples.

[19]For arbitrary bounded functions $P(p,q)$ and $Q(p,q)$, the line integral along the graph of a function S is

$$\int_{\text{Graph of S}} Q(p,q)dq + P(p,q)dp = \int_a^b Q(p,S(p))dS + \int_a^b P(p,S(p))dp$$

whenever each of the integrals on the right hand side exists. See Theorem 10-33, page 276, in Apostol (1957).

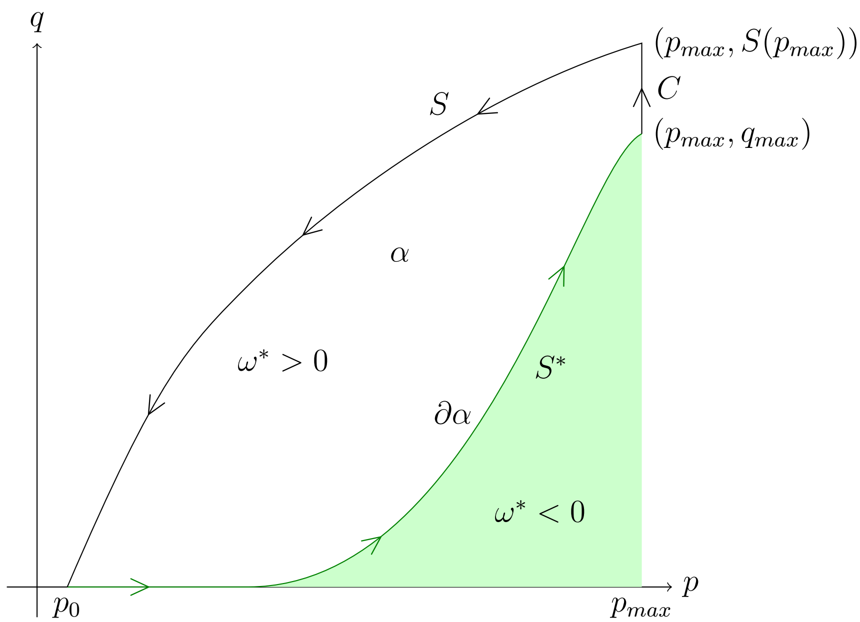

FIGURE 5.— Green's Theorem and the Optimality of S^*. The sign of ω^* is negative in the shaded area and positive elsewhere. Green's Theorem and the definition of q_{max} imply that $I(S^*) - I(S)$ is greater than the double integral of ω^* in the region α. Because ω^* is positive in α it follows that the difference $I(S^*) - I(S)$ is positive.

We have

$$0 \leq \iint_{\alpha} \omega^*(p,q)dpdq \tag{5.3a}$$

$$= \iint_{\alpha} Q_p - P_q dpdq \tag{5.3b}$$

$$= \oint_{\partial\alpha} Qdq + Pdp \tag{5.3c}$$

$$= I(S^*) - I(S) + \int_C Qdq + Pdp \tag{5.3d}$$

$$\leq I(S^*) - I(S) \tag{5.3e}$$

Inequality (5.3a) follows from the (SGN) condition: ω^* is positive at all those points that are above the graph of S^*.

Equality (5.3b) follows from the (AC) condition: Q_p and P_q exist almost everywhere and satisfy (almost everywhere) $\omega^* = Q_p - P_q$.

Equality (5.3c) follows from Green's Theorem (see Appendix B and in particular Lemma 4). Equality (5.3d) holds because outside the interval of prices $[p_0, p_{max}]$, S and S^* are identical. To get inequality (5.3e) we make use of the boundary condition $S^*(p_{max}) = q_{max}$. Indeed, since C is a vertical curve, "$dp = 0$" along C. Hence

$$\int_C Qdq + Pdp = \int_C Qdq = \int_{q_{max}}^{S(p_{max})} u(p_{max}, q)dq \leq 0$$

where the inequality is from the definition of q_{max} (see (4.2)). We conclude that $I(S^*) > I(S)$ for every feasible supply function with a graph that lies above the graph of S^*.

We now consider an aggregate feasible supply function S as shown in Figure (6). Let α be the set of points that are above the graph S^* and below the graph of S, and let β be the set of points that are below the graph of S^* and above the graph of S. Let $\partial\alpha$ and $\partial\beta$ be the counterclockwise oriented curves that forms the boundary of α and β, respectively. Let $C = \{(p_{max}, q) : S(p_{max}) \leq q \leq q_{max}\}$.

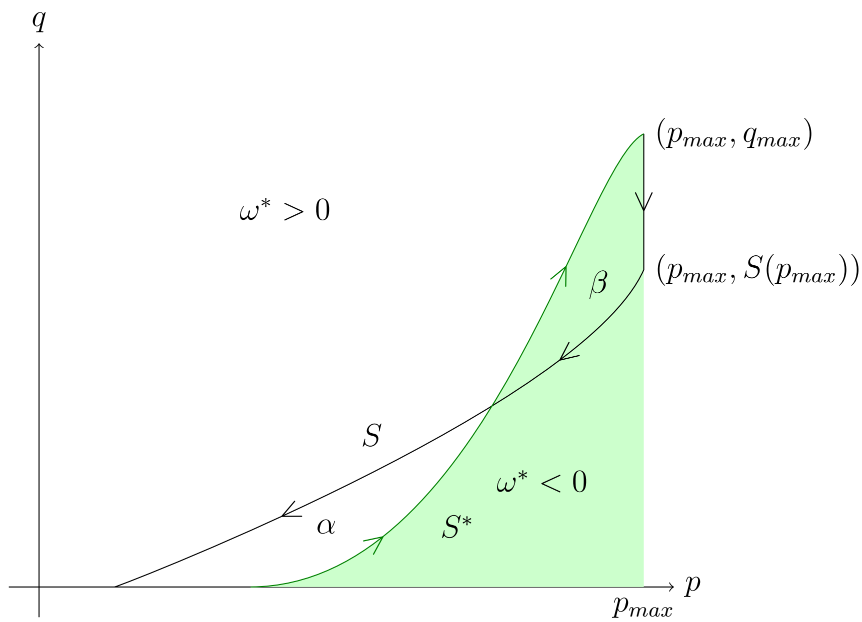

FIGURE 6.— Green's Theorem and the optimality of S^*. Green's Theorem and the definition of q_{max} imply that $I(S^*) - I(S)$ is greater than the double integral of ω^* in the region α minus the double integral of ω^* in the region β. Because ω^* is positive in α and negative in β, it follows that the difference $I(S^*) - I(S)$ is positive.

Similar arguments to those made in (5.3) yield

$$0 \le \iint_{\alpha} \omega^*(p,q)dpdq - \iint_{\beta} \omega^*(p,q)dpdq \tag{5.4a}$$

$$= \iint_{\alpha} Q_p - P_q dpdq - \iint_{\beta} Q_p - P_q dpdq \tag{5.4b}$$

$$= \oint_{\partial\alpha} Qdq + Pdp - \oint_{\partial\beta} Qdq + Pdp \tag{5.4c}$$

$$= I(S^*) - I(S) - \int_{C} Qdq + Pdp \tag{5.4d}$$

$$\le I(S^*) - I(S) \tag{5.4e}$$

Inequality (5.4a) follows from the (SGN) condition: ω^* is positive (resp. negative) at all those points that are above (resp. below) the graph of S^*. Equality (5.4b) follows from the (AC) condition: Q_p and P_q exist almost everywhere and satisfy (almost everywhere) $\omega^* = Q_p - P_q$. Equality (5.4c) follows from Green's Theorem. Equality (5.4d) holds because of the definition of the vertical plane curve C. Note that we have a minus sign before the line integral along C because we traverse C (or more generally $\partial\beta$) in the opposite direction. Inequality (5.4e) is a consequence of the boundary condition $S^*(p_{max}) = q_{max}$: Because C is a vertical curve, "$dp = 0$" along C and hence

$$\int_C Qdq + Pdp = \int_C Qdq = \int_{S(p_{max})}^{q_{max}} u(p_{max}, q)dq \geq 0$$

where the inequality follows from the definition of q_{max} (see (4.2)).

In a similar way we show that the payoff associated with S^* is greater than the payoff associated with any feasible aggregate supply function. *Q.E.D.*

Note that the dynamic programming approach and Miele's approch agree, as expected: above the ask price p^*_{ask}, Miele's function is zero along the graph of S^* (see (5.2)). Whereas the Bellman Equation did not give us information off the path, examining the sign of Miele's function off the path tells us whether the extremum we have identified is a minimum or a maximum of the objective.

6. COMPETITION FOR PRECEDENCE AND NON-EXISTENCE OF EQUILIBRIUM

As in the previous section, S^* and p^*_{ask} denote the solution to free boundary problem (4.1), and S^*/n is the candidate to be a symmetric Nash equilibrium. In this section, however, we are interested in economic environments where (i) the candidate for Nash equilibrium converges to the competitive equilibrium, however, (ii) the (SGN) condition is violated in a manner that implies the existence of profitable deviations.

The violation of the (SGN) condition we examine is related to the competition for precedence among liquidity suppliers. It is perhaps useful to recall what precedence is. Typically, the active trader's order is not large enough to be matched with all the offers in the book. Offers are therefore prioritized in a queue, and the active trader's order is matched with offers based on their place in the queue.

The place in the queue is determined by priority rules. The primary precedence rule is the price priority rule: offers with lower offering prices always get higher precedence. If there are multiple offers with the same offering price, then a secondary rule (usually time) determines the place in the queue. In our model, there are price levels with multiple offers. However, because individual supply functions are continuous, those offers are for infinitesimal number of shares. If the incoming market order walks the book up to the price p_0, how we allocate the last fraction of the order among all those infinitesimal offers at p_0 would not change the total payoff of the liquidity suppliers.[20] Hence, secondary precedence rules play no role in our

[20]Moreover, in our model, offers can be posted at any price level. Thus every

model.

The economic environment, in our paper, is summarized by the profitability, $u(p,q)$, of an offer of an infinitesimal size, where p is the offering price and q is the number of shares that have higher precedence i.e., q is the place in the queue. For a small ϵ, $u(p,q+\epsilon)-u(p,q)\approx u_q(p,q)\epsilon$. Hence, we can think of u_q as a measure of the sensitivity of profits to the place in the queue. For example, if $u_q = 0$ then precedence is irrelevant. On the other hand, if precedence is a virtue then profits should fall as the offer is further back in the queue; i.e., $u_q < 0$.

The marginal rate of substitution between the offering price and the place in the queue, u_p/u_q, measures the tradeoff between potential higher revenue (higher offering price) and lower precedence. The violation of the (SGN) condition we are interested in is related to increasing marginal rate of substitution (IMRS):

(IMRS): *The marginal rate of substitution, u_p/u_q, is increasing in q*

The marginal rate of substitution is the negative of the slope of the level curves of u. If the profitability function satisfies the (IMRS) condition, then the slopes of the level curves are decreasing as we increase q (see Figure 7). Alternatively, at higher offering prices, the level curves get vertically closer; i.e., as we increase prices, the sensitivity of the profitability function to a small change in the place in

liquidity supplier can costlessly undercut by placing the offer below, but arbitrary close to, p_0; thereby invoking the primary priority rule.

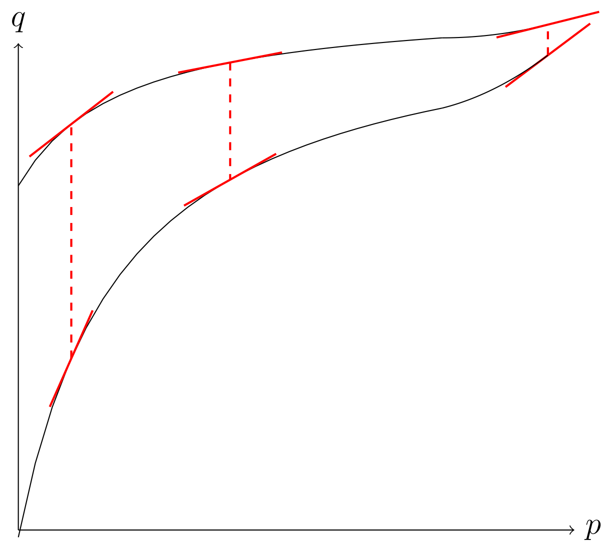

FIGURE 7.— The figure shows two level curves of a profitability function that satisfies the (IMRS) condition. The slopes of the level curves are decreasing as we increase q, and hence the level curves get closer: a small change in the place in the queue results in a larger change in profitability.

the queue is increasing. Intuitively, if the profitability function satisfies the (IMRS) condition then precedence is even more important further back in the book.

Perhaps the most intuitive illustration of a an economic environments in which precedence is even more important further back is the News Traders environment (see Section 2.1). Indeed, the size of the news traders' order is unbounded whereas the size of the liquidity traders' order is bounded. Thus, further back in the book, the risk of trading with informed traders is very high.[21]

[21]Below we verify that in Glosten's uniform example the (IMRS) condition holds.

To see the relation between the (IMRS) condition and violations of the (SGN) condition, we divide each side of (5.2) by u_q, and get that for every $p \in (p^*_{ask}, p_{max})$

$$\frac{\omega^*(p,q)}{u_q(p,q)} = \frac{u_p(p,q)}{u_q(p,q)} - \frac{u_p(p,S^*(p))}{u_q(p,S^*(p))} \tag{6.1}$$

Thus, the SGN condition is violated if there exists $p_0 \in [p^*_{ask}, p_{max}]$, such that at $(p_0, S^*(p_0))$ (i) precedence is a virtue and (ii) the (IMRS) condition holds; i.e., $u_q(p_0, S(p_0)) < 0$ and $\frac{\partial}{\partial q}\frac{u_p(p,q)}{u_q(p,q)}\Big|_{(p_0,S(p_0))} > 0$. Indeed, it follows from (IMRS) that for q slightly greater (resp. smaller) than $S^*(p_0)$, the right hand side of (6.1) is positive (resp. negative). Hence the same is true for the right hand side of (6.1). Because precedence is a virtue, at least in a small neighborhood of $(p_0, S^*(p_0))$, the sign of ω^* is negative above the graph of S^* and positive below it.

To summarize, we expect a profile of individual supply functions to fail to be an equilibrium if the graph of the aggregate supply function, associated with the profile, goes through a region in the pq-plane where the profitability function satisfies the (IMRS) condition. A priori, we don't know the relevance of the (IMRS) condition. However, we know that any aggregate supply function intercepts the p-axis (at the ask price). We can therefore state the following non-existence result.

LEMMA 2 *If the profitability function, u, is twice continuously differentiable in a region that contains the horizontal segment $q = 0$, $0 \le p < p_{max}$, and for each $p_0 \in (0, p_{\max})$, the partial derivatives* *satisfy* $u_q(p_0, 0) < 0$ *and*

$$\frac{\partial}{\partial q}\frac{u_p(p,q)}{u_q(p,q)}\bigg|_{(p_0,0)} > 0$$

then the only piecewise differentiable Nash equilibrium that may exist is $S_i \equiv 0$. *In particular, the candidate we derived from the Bellman equation is not an equilibrium.*

Similar to the proof of Theorem 5.1, we use Green's theorem to prove the lemma. The proof is provided in Appendix A. The lemma does not rule out the possibility that there is a trivial equilibrium in which liquidity suppliers don't post offers. Such an equilibrium may exist if p_{max} is so small that shares can only be offered at a loss. In that case, every liquidity supplier indeed finds it optimal not to offer shares.

We conclude this section with two examples that illustrate the failure of the Bellman equation to characterize the equilibrium.

Glosten's Uniform Example:

Let $0 \le p_{max} < L$.[22] To see that the conditions in Lemma 2 hold, we check

$$u_q(p, 0) = -\mu/2L < 0$$

and

$$\frac{\partial}{\partial q}\frac{u_p(p,q)}{u_q(p,q)}\bigg|_{(p_0,0)} = \frac{1}{p_0} > 0$$

[22] In the uniform example, offers posted at prices greater than L will never be executed. Thus, we don't lose generality if we assume that $P_{max} < L$.

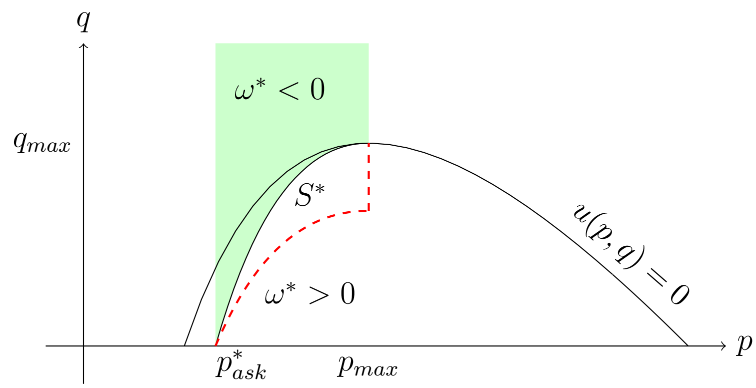

FIGURE 8.— Miele's Function in The Uniform Example. The sign of ω^* is negative in the shaded area and positive elsewhere. The figure shows that the (SGN) condition is violated: for prices greater than p^*_{ask}, the sign is negative above the graph of S^* and positive below it. The dashed graph illustrates a profitable deviation (there are many other). In this deviation, the n-th liquidity supplier does not offer shares at any price below p_{max}. Instead, at $p_{\max}$, a discrete offer with of a size of $\int_0^{p_{max}} s^*(p)dp$ is placed. The number of liquidity suppliers in this example is $n = 3$, and, as in Figures (1) and (2), the parameters are $\mu = 0.5$ and $L = 1$. The exogenous maximum price at which liquidity suppliers can post offers is $p_{max} = 1/\sqrt{3}$.

Thus, we conclude that S^*/n does not form a symmetric equilibrium. Figure 1 illustrates that this economic environment is regular, Figure 2 illustrates the convergence of the candidates we derived from the Bellman equation to the competitive equilibrium, and Figure 8 shows that the SGN condition is indeed violated. An example of a profitable deviation (there are many) is to offer no shares up to p_{max} and at p_{max} to place a discrete offer with a size of q_{max}/n shares. The dashed graph in Figure 8 is the aggregate supply function associated with this deviation.[23]

[23]Formally, we required a feasible supply function to be continuous. Clearly, we can approximate the profitable deviation using continuous, and hence feasible,

Figure 8 implies that S^*/n is an argument of minimum rather than an argument of maximum. Formally, let $\mathcal{S}_i$ denote the class of feasible supply functions with $S_i(p^*_{ask}) = 0$ and $S_i(p_{max}) = q_{max}/n$. Then, S^*/n minimizes the ex-ante expected profit of the n-liquidity supplier in the class of feasible strategies $\mathcal{S}_i$.

The Quadratic-Exponential Example:

The quadratic-exponential example is a special case of the exponential environment (see Section 2.2). This example demonstrates that the results in this paper are in conflict with BMR.

We let $\gamma\sigma^2 = 1$, and set

$$\begin{aligned} \widetilde{v}_s &= 2a\widetilde{u}_1^2\widetilde{u}_2 \\ \widetilde{I} &= \widetilde{v}_s - \widetilde{u}_1 \end{aligned}$$

where $\widetilde{u}_1$ and $\widetilde{u}_2$ are two independent standard uniform random variables, and a is a parameter.

The summary statistics, $\widetilde{\theta} = \widetilde{v}_s - \widetilde{I} = \widetilde{u}_1$, is a standard uniform random variable, and the conditional expectation is

$$\begin{aligned} v(\theta) &= E[\widetilde{v}|\widetilde{\theta} = \theta] = E[\widetilde{v}_s|\widetilde{\theta} = \theta] \\ &= E[2a\widetilde{u}_1^2\widetilde{u}_2|\widetilde{u}_1 = \theta] = 2a\theta^2 E[\widetilde{u}_2|\widetilde{u}_1 = \theta] = a\theta^2 \end{aligned}$$

The first equality above is the definition of the function $v(\theta)$. The second equality comes from $\widetilde{v} = \widetilde{v}_s + \widetilde{\epsilon}$ and the assumption that $\widetilde{\epsilon}$ is

strategies.

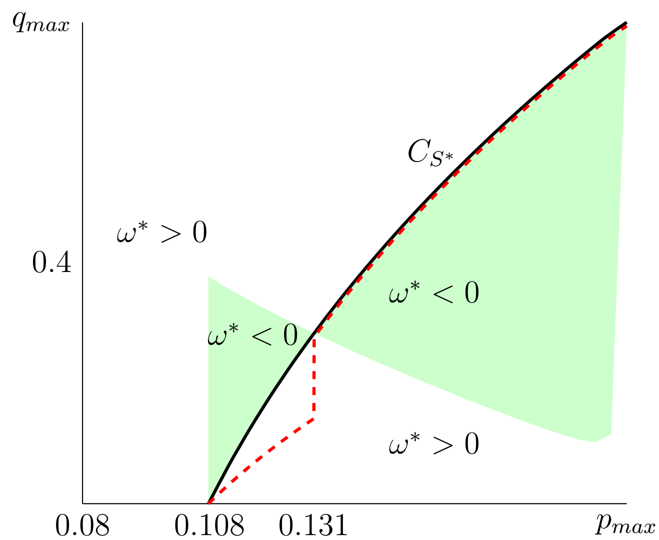

FIGURE 9.— A Profitable Deviation in the Quadratic-Exponential Example. This figure shows a special case of the exponential environment, where $\widetilde{\theta}$ is a standard uniform random variable and $v(\widetilde{\theta}) = a\widetilde{\theta}^2$. The parameters are $n = 2$, $a = 0.2$ and $p_{max} = 0.2$ (which implies $q_{max} = 0.8$). The region where the sign of ω^* is negative is shaded. The dotted graph illustrates one profitable deviation (there are many others). The profitable deviation is not to offer any share up to the price 0.131, and at that price to put a discrete offer with a size of $\int_0^{0.131} s^*(p)dp$ shares. For prices greater than 0.13, the deviation is identical to s^*.

independent of $\widetilde{v}_s$ and $\widetilde{I}$, and hence $\widetilde{\epsilon}$ is also independent of $\widetilde{\theta}$. The last equality is because $\widetilde{u}_1$ and $\widetilde{u}_2$ are independent and $E\widetilde{u}_2 = 1/2$.

In this economic environment, offers at prices greater than $\bar{\theta} = 1$ are never filled up. Thus, we don't lose generality it we assume $p_{max} < 1$.

LEMMA 3 *If the parameter a is in $(0, 1/2)$, then the conditions in Lemma 2 are satisfied.*

The proof of this lemma is in Appendix A. We conclude that also in this example, the Bellman equation does not characterize the equilibrium.

Another way to look at the failure of the equilibrium in the quadratic-exponential example is to note that the conditional expectation is quadratic. Not surprisingly, Miele's function inherits the quadratic property. Consequently, Miele's function changes signs more than once, and the (SGN) condition is violated.

Figure 9 shows an example where we solve (4.1) numerically. In this example, we use the endogenous maximum price implied from BMR (i.e., $p_{max} = v(\bar{\theta})$). The example illustrates how easily we can construct a profitable deviation.[24]

In Appendix C we show that the Bellman equation characterizes the "equilibrium" in BMR and that the quadratic-exponential example satisfies all the technical conditions in BMR.

7. CONCLUSION

We have developed a model of imperfect competition in liquidity provision and made four contributions to the literature. First, we

[24]BMR note that the differential equation has singularities at the boundary. We add that when we take the exogenous price to be strictly smaller than $v(\bar{\theta})$, there is no singularity and one can easily solve (4.1). To solve the equation in Figure 9, we take p_{max} to be arbitrary close to $v(\bar{\theta})$, and solve the differential equation backward until we reach the ask price. Alternatively, we could use a shooting method: guess the ask price and solve the differential equation (4.1) forward and accept the solution if $u(\bar{\theta}, S(\bar{\theta})) = 0$.

showed how to use standard dynamic programming to characterize the equilibrium. Second, we showed how to use Green's theorem to verify the equilibrium.

Third, we found that the intuition that underlies the break-even conditions is incomplete. Explicitly, the literature assumes that the break-even conditions are the outcome of the interaction between many liquidity suppliers who compete their profits away. Implicitly, the literature assumes that there *exists* an equilibrium with a finite number of liquidity suppliers that converges, as the number of liquidity suppliers increases, to an equilibrium that satisfies the break-even conditions (e.g. see Assumption 3 in Glosten (1994)). However, we found that in economic environments where competition for precedence among the liquidity suppliers is more relevant further back in the book (i.e. "behind the market"), existence of equilibrium is an issue. Moreover, the candidate we derived using the Bellman equation converges to the competitive equilibrium (i.e., the break-even conditions are satisfied), even though the candidate characterizes an argument of minimum of the liquidity suppliers' objective function. Thus, our paper challenges both the implicit and explicit assumptions that underlie the break-even conditions.

Finally, we presented an economic environment in which (*i*) all the technical conditions stated in BMR are satisfied, and (*ii*) competition for precedence is more relevant behind the market. We then showed a profitable deviation from the "equilbrium" computed in BMR.

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APPENDIX A: MISCELLANEOUS PROOFS

PROOF OF LEMMA 1: To emphasize that the only random element, from the point of view of the active trader, is the noise term, $\widetilde{\epsilon}$, we omit the tilde over I, v_s, and θ. The active trader sees the supply function S, which is a positive and non-decreasing function, and can infer the total cost of q shares. Formally, we denote the left-continuous inverse of the supply function by S^{-1}; i.e., $S^{-1}(q) = \sup\{p : S(p) < q\}$.[25] Thus, $S^{-1}(q) = p$ is the price of the last fraction of the order and $\int_0^q S^{-1}(y)dy$ is the cost of the total order. The cost of the total order is convex in the size of the order because S^{-1} is non-decreasing.

The active trader's buys $q \geq 0$ that maximizes[26]

$$E - \exp(-\gamma \widetilde{W})$$

where $\widetilde{W} = (q + I)(v_s + \widetilde{\epsilon}) - \int_0^q S^{-1}(y)dy$ is a normal random variable. Because of the exponential utility function, the investor's problem is equivalent to the concave problem:

$$\begin{aligned}
&\max_{q \geq 0}\ E\widetilde{W} - \frac{\gamma}{2} Var(\widetilde{w}) \\
=&\max_{q \geq 0}\ (q + I)v_s - \int_0^q S^{-1}(y)dy - \gamma\frac{\sigma^2}{2}(q + I)^2 \\
=&Iv_s - \gamma\frac{\sigma^2}{2}I^2 + \max_{q \geq 0}\ qv_s - \int_0^q S^{-1}(y)dy - \gamma\frac{\sigma^2}{2}(q^2 + 2qI) \\
=&Iv_s - \gamma\frac{\sigma^2}{2}I^2 + \max_{q \geq 0}\ \theta q - \gamma\frac{\sigma^2}{2}q^2 - \int_0^q S^{-1}(y)dy
\end{aligned}$$

[25]The supply function S may have intervals of constancy that correspond to holes in the book; i.e., price intervals in which offers are not posted. We can restrict the domain of S to exclude the intervals of constancy in a manner that S is left-continuous and S^{-1}, as we defined, is then indeed the inverse of S.

[26]It may be optimal for the active trader to sell, and in that case, the optimal buying size would be zero.

If the optimal size turns out to be strictly positive, then it equals $\sup\{q : \theta - \gamma\sigma^2 q - S^{-1}(q) > 0\}$, or equivalently, the order walks up the book as long as $\theta > \gamma\sigma^2 S(p) + p$.[27] *Q.E.D.*

Proof of Lemma 2: Let $(S_1, \ldots, S_n)$ be a profile of feasible strategies, and let $S = \sum_i S_i$ denote the aggregate supply function associated with the profile. Assume that $S \neq 0$, and that for each i, there is a piecewise continuous function s_i such that $S_i(p) = \int_0^p s_i(x)dx$. Our goal is to show that the profile of strategies does not form a Nash equilibrium.

Let $p_{ask} = \min\{p, S(p) > 0\}$ denote the ask price associated with the profile. The assumption $S \neq 0$ implies $p_{ask} < p_{max}$. Let $s_{-i} = \sum_{j \neq i} s_j$, and let

$$\omega(p, q) \equiv u_p(p, q) + u_q(p, q)s_{-i}$$

Under the assumptions in the Lemma, ω is continuous in a neighborhood that contains the vertical segment $q = 0$, $0 < p < p_{\max}$. Hence, one and only one of the following holds: (i) there exists an $\epsilon > 0$ such that for every $p \in (p_{ask}, p_{ask} + \epsilon)$, $\omega(p, S(p)) \neq 0$, or (ii) there exists an $\epsilon > 0$ such that for every $p \in (p_{ask}, p_{ask}+\epsilon)$, $\omega(p, S(p)) = 0$. We consider each case separately.

(i) Say $\omega(p, S(p)) \neq 0$ on $(p_{ask}, p_{ask} + \epsilon)$. Because ω is continuous, there is a narrow band B that surrounds the graph of S such that the sign of ω does not change in the band. We first assume the sign of ω in positive in the band B.

Because $p_{ask} < p_{max}$, there exists at least one individual supply function, say the i-th one, with $S_i(p) > 0$ for $p \in (p_{ask}, p_{ask} + \epsilon)$. Let now $\bar{S}_i$ be a feasible supply function such that $\bar{S}_i \leq S_i$ and all the points between the graphs of the

[27] The objective function of the concave problem may not be smooth because S^{-1} may not be continuous: intervals of constancy of S correspond to jumps of S^{-1}. The argument we invoke is that the optimal size is greater than any size at which the derivative of the objective function is strictly positive. And this is true because the problem is concave.

aggregate supply functions $\bar{S}_i + S_{-i}$ and $S_i + S_{-i}$ are in the band B. In particular, $\bar{S}_i = S_i$ outside the interval $(p_{ask} < p < p_{ask} + \epsilon)$.

Let α denote all the points that are below the graph of S and above the graph of $\bar{S} \equiv \bar{S}_i + S_{-i}$. Let $\partial\alpha$ denote the counterclockwise oriented curve that forms the boundary of α. In particular, ω is strictly positive in α (because α is in the band B).

We insert the constraint (1.2) into the objective (1.1) and express the payoffs of S_i in terms of the aggregate supply function S

$$\int_0^{p_{max}} u(p, S(p))dS(p) - \int_0^{p_{max}} u(p, S(p))s_{-i}(p)dp$$

and similarly, the payoff of $\bar{S}_i$ is expressed in terms of the aggregate supply function $\bar{S}$:

$$\int_0^{p_{max}} u(p, \bar{S}(p))d\bar{S}(p) - \int_0^{p_{max}} u(p, \bar{S}(p))s_{-i}(p)dp$$

The payoffs associated with $\bar{S}_i$ minus the payoff associated with S_i is then the line integral

$$\oint_{\partial\alpha} u(p, q)dq - u(p, q)s_{-i}(p)dp = \iint_{\alpha} \omega(p, q)dqdp > 0$$

where the equality is Green's theorem and the inequality is because ω is positive in α. This proves that $\bar{S}_i$ is a profitable deviation.

The case $\omega < 0$ in the band B is analogous. This time, however, the deviation involves offering more shares at lower prices (i.e., the graph of the deviation lies above the graph of S_i).

(ii) We now assume $\omega(p, S(p)) = 0$ for all $p \in (p_{ask}, p_{ask} + \epsilon)$. We are interested in the sign of ω below the graph of S. Because the partial derivatives of u are continuous in a region that contains $q = 0$, $0 \le p < p_{max}$, there exists a small

region A, bounded from above by the graph of S, such that in the region the partial derivatives satisfy the inequalities stated in the lemma. That is, $u_q < 0$ and u satisfies the (IMRS) condition. To see now that ω is positive in the region A, we note that $\omega(p, S(p)) = 0$ implies

$$u_p(p, S(p))/u_q(p, S(p)) + s_{-i}(p) = 0$$

Thus, for $(p, q) \in A$ (i.e. $q < S(p)$), we have

$$\omega(p, q) = u_q(p, q) \left(\frac{u_p(p, q)}{u_q(p, q)} + s_{-i}(p) \right) > u_q(p, q) \left(\frac{u_p(p, S(p))}{u_q(p, S(p))} + s_{-i}(p) \right) = 0$$

where the inequality is because $u_q < 0$, u_p/u_q is increasing in q, and $q < S(q)$. Thus, below the graph of S the sign of ω is positive. It follows from Green's theorem that any feasible strategy, $\bar{S}_i$, is a profitable deviation provided that the aggregate supply function $\bar{S} = \bar{S}_i + S_{-i}$ is identical to S outside the interval $(p_{ask}, p_{ask} + \epsilon_2)$ and smaller than S in the interval. *Q.E.D.*

Proof of Lemma 3: We need to check that the profitably function satisfies the three conditions stated in Lemma 2. Let $p_{max} < 1$. The profitability function is given (2.2), where (i) $\bar{\theta} = 1$, (ii) $\gamma\sigma^2 = 1$, (iii) $F(\theta) = \theta$, and (iv) $v(\theta) = a\theta^2$.

Because $p_{max} < 1$, there is a small region in the pq-plane that contains the horizontal segment $q = 0$, $0 \leq p < p_{max}$, such that in the region the profitability function is

$$u(p, q) = p(1 - (p + q)) - \frac{a}{3}\left(1 - (p + q)^3\right)$$

and in particular u is smooth there, which is the first condition we had to verify.

Next, we need to check that $u_q(p_0, 0) < 0$. For $p_0 < p_{max} < 1$ we have

$$u_q(p_0, 0) = -p + ap_0^2 < 0$$

where the inequality is because $a < 1/2$ and $p_0 < 1$.

Finally, we need to check the (IMRS) condition holds. We have

$$u_p(p,q) = (1 - F(q+p)) + F'(q+p)(v(q+p) - p)$$

and

$$u_q(p,q) = F'(q+p)(v(q+p) - p)$$

and the marginal rate of substitution is

$$\frac{u_p(p,q)}{u_q(p,q)} = \frac{(1 - F(q+p))}{F'(q+p)} \frac{1}{(v(q+p) - p)} + 1 = (1 - q - p)\frac{1}{(v(q+p) - p)} + 1$$

where for the second equality we used $F(\theta) = \theta$. Next,

$$\begin{aligned}\frac{\partial}{\partial q}\frac{u_p(p,q)}{u_q(p,q)} &= -\frac{1}{(v(q+p) - p)} - (1 - q - p)\frac{v'(q+p)}{(v(q+p) - p)^2} \\ &= \frac{-a(q+p)^2 + p - (1 - q - p)2a(q+p)}{(v(q+p) - p)^2}\end{aligned}$$

where for the second equality we used $v(\theta) = a\theta^2$. Thus, for every $p_0 \in (0,1)$, we have

$$\left.\frac{\partial}{\partial q}\frac{u_p(p,q)}{u_q(p,q)}\right|_{(p_0,0)} = \frac{-ap^2 + p - (1-p)2ap}{(v(q+p) - p)^2} = \frac{ap^2 + p - 2ap}{(v(q+p) - p)^2} > 0$$

where the last equality is because $a < 1/2$. *Q.E.D.*

APPENDIX B: GREEN'S THEOREM

Green's theorem (or Green's identity) relates a double integral over a region with a line integral over the boundary of the region:

$$\text{(B.1)} \qquad \oint_{\partial\alpha} Qdq + Pdp = \iint_{\alpha} (Q_p - P_q)\, dqdp$$

where α is a region in the pq-plane and $\partial\alpha$ is the counterclockwise oriented closed curve that forms its boundary.[28] In Theorem 5.1, we use Green's identity with the functions

$$\text{(B.2)}\qquad \begin{aligned} Q(p,q) &= u(p,q) \\ P(p,q) &= -u(p,q)\frac{(n-1)}{n}s^*(p) \end{aligned}$$

In particular, the functions P and Q don't satisfy the standard assumptions imposed in calculus textbooks; i.e., in standard textbooks the functions Q and P are continuously differentiable. In our paper, $s^*(p)$ has a discontinuity at $p = p^*_{ask}$ (see (5.1)), and hence the function P has a vertical curve of discontinuity at $p = p^*_{ask}$. Also, whenever the random variables that underlie the economic environment are finite, the profitability function, u, may not be continuously differentiable everywhere. That said, the continuous differentiability assumption is not necessary, at least when the region is not too complicated. Also we will show that because the points of discontinuity of P are all on a vertical curve (i.e. $p = p^*_{ask}$), Green's identity can be used in Theorem 5.1 .

The regions we consider in Theorem 5.1 consist of points that lie between the graphs of two monotone functions. These regions belong to two special classes of regions: q-simple and p-simple regions.[29] The simplicity in the sense that we can express the double integral over a region as an iterated integral and moreover we can interchange the order of integration.

We first note that the Green's identity is equivalent to the two identities

$$\text{(B.3)}\qquad \oint_{\partial\alpha} P dp = -\iint_{\alpha} P_q dq dp$$

and

$$\text{(B.4)}\qquad \oint_{\partial\alpha} Q dq = \iint_{\alpha} Q_p dq dp$$

[28]More accurately, the orientation of the curve is such that as we traverse the curve, the region is to our left.

[29]Some text books call these regions type I and type II regions or x-simple and y-simple regions.

Indeed, we can take in (B.1) $Q \equiv 0$ (repc. $P \equiv 0$) to get (B.3) (resp. (B.4)). Also, if (B.3) and (B.4) hold, then by adding them up we get (B.1).

THEOREM B.1 *Let the region α be given by the inequalities $a \leq p \leq b$ and $q_1(p) \leq q \leq q_2(p)$ (i.e. α is a q-simple region), and let $\partial\alpha$ be its, counterclockwise oriented, boundary. If (i) $P(p,q)$ is continuous on $\partial\alpha$ and α except at points that lie on a finite number of vertical curves, and (ii) for almost every $p \in (a,b)$, the function $q \to P(p,q)$ is absolutely continuous in the interval $[q_1(p), q_2(p)]$, then* (B.3) *holds.*

Reid (1941) proves the theorem for more general regions, however under the condition that the function $P(p,q)$ is continuous. While we relax the continuity assumption, we note that it is crucial that the curves of discontinuity of $P(p,q)$ are vertical. In fact, conditions (i) and (ii) above would be inconsistent if the curves of discontinuity are arbitrary.

PROOF: For almost every $p \in (a,b)$ the function $q \to P(p,q)$ is absolutely continuous in $[q_1(p), q_2(p)]$. Hence, for almost every $p \in (a,b)$

$$P(p, q_2(p)) - P(p, q_1(p)) = \int_{q_1(p)}^{q_2(p)} P_q(p,q)dq \tag{B.5}$$

Because α is q-simple, the boundary is $\partial\alpha = C_1 + C_2 - C_3 - C4$ where $C_1 = \{(p, q_1(p)) : a \leq p \leq b\}$, $C_2 = \{(b,q) : q_1(b) \leq q \leq q_2(b)\}$, $C_3 = \{(p, q_2(p)) : a \leq p \leq b\}$, and $C_4 = \{(a,q) : q_1(a) \leq q \leq q_2(a)\}$. We put a minus sign before C_3 and C_4 because we traverse $\partial\alpha$ in the counterclockwise direction.

We have

$$\begin{aligned}\oint_{\partial\alpha} Pdp &= \int_{C_1} Pdp + \int_{C_2} Pdp - \int_{C_3} Pdp - \int_{C_4} Pdp \\ &= \int_a^b P(p, q_1(p))dp + \int_b^a P(p, q_2(p))dp \\ &= -\int_a^b P(p, q_2(p)) - P(p, q_1(p))dp \\ &= -\int_a^b \int_{q_1(p)}^{q_2(p)} P_q dqdp = \iint\limits_{\alpha} -P_q dqdp\end{aligned}$$

where the first equality follows from the definition of $\partial\alpha$, the second equality is true because the line integral is zero along the vertical curves C_2 and C_4 (where "$dp = 0$"), the third equality is immediate, and the fourth equality is justified by (B.5). *Q.E.D.*

THEOREM B.2 *Let the region α be given by the inequalities $a \leq q \leq b$ and $p_1(q) \leq p \leq p_2(q)$ (i.e. α is p-simple), and let $\partial\alpha$ be the boundary (counterclockwise oriented). If (i) $Q(p, q)$ is continuous on $\partial\alpha$ and α except at points that lie on a finite number of horizontal curves, and (ii) for almost every $q \in (a, b)$, the function $p \to Q(p, q)$ is absolutely continuous in the interval $[p_1(q), p_2(q)]$, then* (B.4) *holds.*

The proof mirrors the proof of Theorem (B.1) and we omit it.

LEMMA 4 *Assume (AC) holds, and let α be a region that consists of those points that lie between the graphs of two feasible supply functions. Then*

$$\oint\limits_{\partial\alpha} u(p, q)dq - u(p, q)\frac{(n-1)}{n}s^*(p)dp = \iint\limits_{\alpha} \omega^*(p, q)dpdq$$

PROOF: Let P and Q be as in (B.2). We need to show that Green's identity (B.1) holds, or alternatively we need to show that (B.3) and (B.4) hold.

To see that (B.3) holds, we verify the conditions in Theorem B.1. Because α is a region between the graphs of two functions, it is by definition q-simple. Because u

satisfies (AC), it is continuous. Also s^* is continuous everywhere except at a single point, namely p^*_{ask} (see (5.1)). Thus, the function $P(p,q) = -u(p,q)\frac{(n-1)}{n}s^*(p)$ is continuous except at points that lie on a single vertical curve. Also, because u satisfies (AC) and s^* has only one point of discontinuity, also the second condition in Theorem B.1 holds. We conclude that the identity (B.3) is valid.

To see that (B.4) holds, we check the conditions in Theorem B.2. Because α is a region between the graphs of two feasible supply functions (i.e. monotone) it is p-simple. Because $Q(p,q) = u(p,q)$ and u satisfies the (AC) condition, both conditions stated in Theorem B.2 hold, and we conclude that also B.4 holds. *Q.E.D.*

APPENDIX C: THE LINK TO BMR

BMR study only the exponential environment, and in this appendix we relate our result to theirs. We will show that the Bellman equation characterizes the equilibrium computed in BMR. This result is not surprising because the Bellman equation pins down the candidate to be a continuous equilibrium, and BMR's equilibrium is continuous. BMR verify the equilibrium under very mild technical conditions on the distributions of the underlying random variables. In particular, the (SGN) condition we imposed in our verification theorem does not show in BMR (see Propositions 8, 9, and 10 in BMR). This raises questions about the validation of the equilibrium in BMR. We will conclude this appendix with a proof that the quadratic-exponential example satisfies all the technical conditions BMR impose, even though we showed, in Section 6), that no equilibrium exists in this example.

As in BMR, we consider now the exponential environment and assume that the active trader's signal, $\widetilde{v}_s$, and the initial position in the asset, $\widetilde{I}$, have bounded supports. The summary statistics, $\widetilde{\theta} = \widetilde{v}_s - \gamma\sigma^2\widetilde{I}$, and its upper support, $\bar{\theta}$, the conditional expectation, $v(\theta)$, and the upper tail conditional expectation, $v^+(\theta)$, are defined in Section 2.2, and have the same meaning in BMR.[30]

[30]The only difference in notation between our paper and BMR's is the private signal, which we denote here by $\widetilde{v}_s$.

BMR state their assumptions directly on the distribution of the summary statistics. We recall the following assumption from BMR (See page 807 in BMR).

Assumption 1 *The summary statistics, $\widetilde{\theta}$, is a continuous random variable with a bounded support. The conditional expectation function satisfies $v'(\theta) \geq 0$ for all θ in the support of $\widetilde{\theta}$.*

This assumption ensures that $\widetilde{\theta}$ positively correlated with the liquidation value of the asset, and thus also the upper tail expectation is greater than the conditional expectation; i.e., $v^{+}(\theta) > v(\theta)$.

In BMR , the maximum price is endogenous (i.e., liquidity suppliers are not constraint by a maximum price), whereas in our model the maximum price is exogenous. That said, if we set the exogenous price in our model to be BMR's implied endogenous maximum price, then the two models should be comparable. We would see in Lemma 6 below that the endogenous price in BMR is $p_{max} = v(\bar{\theta})$.[31] In the following lemma, we find the total quantity of shares, in our model, that is associated with $p_{max} = v(\bar{\theta})$.

Lemma 5 *If Assumption 1 is satisfied, and $p_{max} = v(\bar{\theta})$, then*

$$q_{max} = \frac{\bar{\theta} - v(\bar{\theta})}{\gamma\sigma^2}$$

Proof of Lemma 5: Assumption 1 implies $v^{+}(\bar{\theta}) = v(\bar{\theta})$. The profitability function, in the exponential environment, is given in (2.2). Hence, we have

$$u\left(v(\bar{\theta}), \frac{\bar{\theta} - v(\bar{\theta})}{\gamma\sigma^2}\right) = 0$$

[31] The implied endogenous maximum price in BMR is the same endogenous maximum price that one finds in the competitive equilibrium. We conjecture that this is true in general; i.e., we conjecture that in every economic environment in which an equilibrium with a continuous aggregate supply function exists, the endogenous maximum price is the same as in the competitive equilibrium.

Assumption 1 also implies that the upper tail expectation function, $v^+(\theta)$, is an increasing function. Hence, $q \to u(v(\bar{\theta}), q)$ is positive in $(0, \frac{\bar{\theta}-v(\bar{\theta})}{\gamma\sigma^2})$, and non-positive for values greater than $\frac{\bar{\theta}-v(\bar{\theta})}{\gamma\sigma^2}$. Thus,

$$\frac{\bar{\theta} - v(\bar{\theta})}{\gamma\sigma^2} = \underset{q\geq 0}{\operatorname{argmax}} \int_0^q u(v(\bar{\theta}), y)dy$$

Q.E.D.

In our model, the strategy space of the liquidity suppliers consists of individual cumulative supply functions. In BMR, on the other hand, the i-th liquidity suppliers is a dealer who posts a *transfer schedules*, T_i, with the interpretation that $T_i(q)$, for a positive quantity q, is the offer price for a total of q shares. Under some technical conditions, that we spell out below, BMR find that the equilibrium is unique, symmetric, convex, and differentiable (see BMR Propositions 8-10). The convexity and differentiability are properties of the equilibrium transfer schedules. When the transfer schedules are convex, BMR model can be interpreted as a model of the limit order book (see the discussion in Section 3.2 in BMR).

The relation between an arbitrary transfer schedule and the corresponding individual supply function is discussed in BMR (see BMR page 815). In particular, when the transfer schedule is differentiable and strictly convex (so the derivative of the transfer schedule is invertible), then the inverse of the derivative of the transfer schedule is the supply function (see Definition 2 in BMR). Intuitively it follows that if the equilibrium transfer schedules are strictly convex and twice differentiable, then the analysis of the Bellman equation, which we carried in Section 4, can be used to characterize the equilibrium (i.e. equation (4.1) in this paper can be used instead of equation (43) in BMR). To make the link rigorous, we have the following.

LEMMA 6 *Assume all the technical assumptions in BMR hold so that equilibrium in BMR exists. In particular, Assumption 1 is satisfied. Let $p_{max} = v(\bar{\theta})$ be the exogenous maximum price in* (4.1). *If a non-decreasing solution to* (4.1) *exists, then it is the aggregate supply function of the equilibrium computed in BMR.*

The idea behind the proof is a change of variable that transoms (4.1) into the equation that BMR use to characterize the equilibrium in their model, namely equation (43) in BMR.

Proof of Lemma 6: BMR use the trading volume, as a function of elements in the support of $\widetilde{\theta}$, to characterize their equilibrium. In Proposition 7 in BMR, they prove that every equilibrium must satisfy equation (43) (see page 820 in BMR).

Equation (43) in BMR captures the bid side as well as the offer side of the book, whereas we only analyzed the offer side of the book. Equation (43), restricted to $[\theta_a^n, \bar{\theta}]$, represents the equilibrium trading volume in BMR when the active trader is a buyer.[32] Thus, our goal is to show that equation (4.1) in our paper, restricted to $[p_{ask}^*, p_{max}]$, is equivalent to equation (43) in BMR, restricted to $[\theta_a^n, \bar{\theta}]$. Both, (4.1) and equation (43) in BMR are free boundary problems. In (4.1), p_{ask}^* has to be determined as part of the solution, and in equation (43) in BMR θ_a^n has to be determined.

We start with an arbitrary p_{max} for which a solution to (4.1), S^* and p_{ask}^*, exists, and the solution is such that S^* is non-decreasing. We define the function Θ via

$$\Theta(p) \equiv \gamma\sigma^2 S^*(p) + p$$

Because the aggregate supply function, S^*, is non-decreasing, the function Θ is strictly increasing, and thus it is invertible. We denote the inverse by Θ^{-1}. Because Θ is strictly increasing, Θ^{-1} is also strictly increasing.

We will show that $q^n(\theta) \equiv S^*(\Theta^{-1}(\theta))$ and $\theta_a^n \equiv \Theta(p_{ask}^*)$ solve equation (43) in BMR. We will then set $p_{max} = v(\bar{\theta})$, which is the implied endogenous maximum price in BMR, and show that the boundary condition stated in Proposition 7 in BMR is satisfied.[33]

[32] See also Section 6.5 in BMR for a discussion of the splitting of equation (43).

[33] To see that q^n, as we defined above, can indeed be thought as the trading volume, we note the following. If the realization of the summary statistics satisfies $\widetilde{\theta} > \theta_a^n$, then the active trader is a buyer. The order walks up the book until the

The definition of the transformation Θ implies that, for $\theta \in (\Theta(p^*_{ask}), \Theta(p_{max}))$, we have

$$\begin{aligned} \dot{q}^n(\theta) &= \dot{S}^*(\Theta^{-1}(\theta))\dot{\Theta}^{-1}(\theta) \\ &= \dot{S}^*(\Theta^{-1}(\theta))\frac{1}{\dot{\Theta}(\Theta^{-1}(\theta))} \\ &= \frac{\dot{S}^*(\Theta^{-1}(\theta))}{\gamma\sigma^2\dot{S}^*(\Theta^{-1}(\theta))+1} \\ &= \frac{1}{\gamma\sigma^2}\left(1+\frac{1}{\gamma\sigma^2\dot{S}^*(\Theta^{-1}(\theta))}\right)^{-1} \qquad \text{(C.1a)} \end{aligned}$$

and the boundary conditions are

$$q^n(\Theta(p^*_{ask})) = 0 \text{ and } q^n(\Theta(p_{max})) = q_{max} \qquad \text{(C.1b)}$$

BMR express the differential equation in (43) in terms of two functions, q^* and q_m. The function q^* is defined in equation (8) in page 808 as

$$q^*(\theta) = \frac{\theta - v(\theta)}{\gamma\sigma^2}$$

and q_m is defined in page 811 as

$$q_m(\theta) = \frac{F(\theta)-1}{\gamma\sigma^2 F'(\theta)} + q^*(\theta)$$

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price level $\Theta^{-1}(\widetilde{\theta})$ and at that price the order is completely filled. Thus, the active trader buys a total of $S^*(\Theta^{-1}(\widetilde{\theta}))$ shares. Hence, what BMR refer to as total volume and express as a function of the summary statistics $\widetilde{\theta}$, is indeed (when the active trader is a buyer) $S^*(\Theta^{-1}(\widetilde{\theta}))$.

Because S^* solves (4.1), for $p \in (p^*_{ask}, p_{max})$ we have

$$\frac{n-1}{n}\dot{S}^*(p) = -\frac{u_p(p,S^*(p))}{u_q(p,S^*(p))} \tag{C.2a}$$

$$= -\frac{1-F\left(\gamma\sigma^2 S^*(p)+p\right)+F'\left(\gamma\sigma^2 S^*(p)+p\right)\left(v\left(\gamma\sigma^2 S^*(p)+p\right)-p\right)}{\gamma\sigma^2 F'\left(\gamma\sigma^2 S^*(p)+p\right)\left(v\left(\gamma\sigma^2 S^*(p)+p\right)-p\right)} \tag{C.2b}$$

$$=\frac{F\left(\Theta(p)\right)-1+F'\left(\Theta(p)\right)\left(p-v\left(\Theta(p)\right)\right)}{\gamma\sigma^2 F'\left(\Theta(p)\right)\left(v\left(\Theta(p)\right)-p\right)} \tag{C.2c}$$

$$=\frac{F\left(\Theta(p)\right)-1+F'\left(\Theta(p)\right)\left(\Theta(p)-v\left(\Theta(p)\right)-\gamma\sigma^2 S^*(p)\right)}{\gamma\sigma^2 F'\left(\Theta(p)\right)\left(\gamma\sigma^2 S^*(p)-\left[\Theta(p)-v\left(\Theta(p)\right)\right]\right)} \tag{C.2d}$$

$$=\frac{F\left(\Theta(p)\right)-1+F'\left(\Theta(p)\right)\left(\Theta(p)-v\left(\Theta(p)\right)-\gamma\sigma^2 q^n(\Theta(p))\right)}{\gamma\sigma^2 F'\left(\Theta(p)\right)\left(\gamma\sigma^2 q^n(\Theta(p))-\left[\Theta(p)-v\left(\Theta(p)\right)\right]\right)} \tag{C.2e}$$

$$=\frac{\frac{F(\Theta(p))-1}{\gamma\sigma^2 F'(\Theta(p))}+\frac{\Theta(p)-v(\Theta(p))}{\gamma\sigma^2}-q^n(\Theta(p))}{\gamma\sigma^2\left(q^n(\Theta(p))-\frac{\Theta(p)-v(\Theta(p))}{\gamma\sigma^2}\right)} \tag{C.2f}$$

$$=\frac{\frac{F(\Theta(p))-1}{\gamma\sigma^2 F'(\Theta(p))}+q^*(\Theta(p))-q^n(\Theta(p))}{\gamma\sigma^2\left(q^n(\Theta(p))-q^*(\Theta(p))\right)} \tag{C.2g}$$

$$=\frac{1}{\gamma\sigma^2}\frac{q_m(\Theta(p))-q^n(\Theta(p))}{q^n(\Theta(p))-q^*(\Theta(p))} \tag{C.2h}$$

In (C.2) above, we used the following transformations. The first equation, (C.2a), is the o.d.e. (4.1). To get (C.2b), we simply insert the partial derivatives of the profitability function, u, (see (2.2)). In (C.2c), we use the definition of the function Θ to replace each occurrence of the term $\gamma\sigma^2 S(p)+p$ with $\Theta(p)$. Also for (C.2d) we use the definition of the function Θ to replace p with $\Theta(p)-\gamma\sigma^2 S(p)$. To get (C.2e), we use the definition of the function q^n to write $S(p)=S(\Theta^{-1}(\Theta(p)))=q^n(\Theta(p))$, and then we replace each occurrence of $S(p)$ with $q^n(\Theta(p))$. We get equality (C.2f) by dividing the numerator and denominator by $\gamma\sigma^2 F'(\Theta(p))$. To get (C.2g), we use the definition of the function q^* to replace each occurrence of $\frac{\Theta(p)-v(\Theta(p))}{\gamma\sigma^2}$ with $q^*(\Theta(p))$. To get the last equality, (C.2h), we use the definition of the function q_m.

Substitute in (C.2), $p=\Theta^{-1}(\theta)$, and multiply each side by $(n-1)/n$ to get

$$\dot{S}(\Theta^{-1}(\theta)) = \frac{1}{\gamma\sigma^2}\frac{n}{n-1}\frac{q_m(\theta) - q^n(\theta)}{q^n(\theta) - q^*(\theta)} \qquad \theta \in (\Theta(p^*_{ask}), \Theta(p_{max}))$$

We insert the above into (C.1a) to conclude that

$$\dot{q}^n(\theta) = \frac{1}{\gamma\sigma^2}\left(1 + \frac{(n-1)\left(q^*(\theta) - q^n(\theta)\right)}{n\left(q^n(\theta) - q_m(\theta)\right)}\right)^{-1}, \theta \in (\Theta(p^*_{ask}), \Theta(p_{max}))$$

which is equation (43) in BMR, restricted to $(\Theta(p^*_{ask}), \Theta(p_{max}))$.

To conclude the proof, we need to verify that the boundary conditions, (C.1b), match the boundary conditions in BMR.

In BMR, the left ("free") boundary, θ^n_a, is chosen so that $q^n(\theta^n_a) = 0$ holds. Because we set $\theta^n_a = \Theta(p^*_{ask})$, the left boundary condition is matched. BMR's right boundary condition is $q^n(\bar{\theta}) = q^*(\bar{\theta})$, whereas in (C.1b) we have $q^n(\Theta(p_{max})) = q_{max}$. To match the right boundary condition in (C.1b) with BMR's, we have to set the exogenous maximum price in (4.1) to $v(\bar{\theta})$. In Lemma 5, we show that

$$q_{max} = \frac{\bar{\theta} - v(\bar{\theta})}{\gamma\sigma^2} = q^*(\bar{\theta})$$

Thus, with $p_{max} = v(\bar{\theta})$, we have $S^*(p_{max}) = q^*(\bar{\theta})$.

The right boundary of the interval $(\Theta(p^*_{ask}), \Theta(p_{max}))$ is then

$$\Theta(p_{max}) = \Theta(\bar{\theta}) = \gamma\sigma^2 S^*(\bar{\theta}) + \bar{\theta} = \bar{\theta}$$

as in BMR. And at that boundary, we have

$$q^n(\bar{\theta}) = q^n(\Theta(p_{max})) = S^*(p_{max}) = q_{max} = q^*(\bar{\theta})$$

which is what BMR impose as the right boundary condition. We conclude that both left and right boundary conditions match.

To summarize, we have shown that when the exogenous maximum price is $v(\bar{\theta})$, a non-decreasing solution, S^*, of (4.1) can be transformed into a solution of equation (43) in BMR, an equation that BMR use to characterize their equilibrium.

Q.E.D.

In Lemma 6, we have shown that the equilibrium in BMR is characterized by the Bellman equation. Recall that we derived (4.1) under the assumption that a continuous equilibrium exists, thus (4.1) represents a necessary condition. To ensure sufficiency, BMR impose the following conditions.

I. [see BMR page 810] Let F be the cumulative and density functions of $\widetilde{\theta}$. Then for every θ in the support of $\widetilde{\theta}$[34]

$$\frac{d}{d\theta}\left(\frac{1-F(\theta)}{F'(\theta)}\right) < 0$$

and

$$\frac{d}{d\theta}\left(\frac{F(\theta)}{F'(\theta)}\right) > 0$$

II. [see BMR page 821] Let $\bar{\theta}$ and $\underline{\theta}$ be the upper and lower support of $\widetilde{\theta}$, respectively.

$$\lim_{\theta\to\bar{\theta}^-}\frac{d}{d\theta}\left(\frac{1-F(\theta)}{F'(\theta)}\right) = -1$$

and

$$\lim_{\theta\to\underline{\theta}^+}\frac{d}{d\theta}\left(\frac{F(\theta)}{F'(\theta)}\right) = 1$$

III. [See BMR page 807] For every θ in the support of $\widetilde{\theta}$, the conditional expectation function satisfies $0 \le v'(\theta) < 1$.

Consider now the quadratic-exponential example from Section 6. We first note that the summary statistics, $\widetilde{\theta}$, is a standard uniform random variable, and hence

[34]BMR actually assume these inequalities hold only in a certain part of the support. In the counter we present, the inequalities hold in the entire support and in particular at the part of the support that is relevant to BMR.

Conditions I and II are satisfied. Because $v(\theta) = a\theta^2$ and $a < 1/2$, we have $v'(\theta) = 2a\theta < 1$ for all θ in the support of $\widetilde{\theta}$. Hence also Condition III is satisfied. However, in Lemma 3 we proved that equilibrium does not exist if the parameter a is in $(0, 1/2)$.

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