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Who Benefits from an Open Limit-Order Book?*

The NYSE opened the limit-order book to off-exchange traders during trading hours. We address the welfare implications of this change in market structure. We model a market similar to the auction that the exchange uses to open the trading day. We consider two different environments. In the first, only the specialist sees the limit-order book, while in the second the information in the book is available to all traders. We compare equilibria and find that traders who demand liquidity are better off when the book is open while liquidity suppliers are better off when the book is closed.

I. Introduction

In 2001, American security markets switched to decimal pricing. Since then, it is argued, the specialists on the New York Stock Exchange (NYSE) and the limit-order traders have been able to change quotes by offering a slightly better price (penny improvement) for a small number of shares. Thus, the inside quotes are no longer a good indicator of market conditions. Addressing the concerns of investors who desire a better look at market depth, the NYSE, as of January 24, 2002, made the limit-order book visible to the public in real time during trading hours. According the NYSE OpenBook Specification, the NYSE disseminates a full view of limit-order book beginning at 7:30 A.M., 2 hours before the market opens. In this paper, we develop a model to address the welfare implications of making the limit-order book visible prior to market opening.

The NYSE begins the trading day at 9:30 A.M. with a single-price call-type auction. "At the opening" market buy and "at the opening" market sell orders that accumulated while the exchange was closed are paired automatically by the Opening Automated Report Service (OARS). The imbalance is presented to the specialist, who then compares it

* I am grateful to Kerry Back for his guidance. I thank Hank Bessembinder, Phil Dybvig, Michel Habib, Eric Hughson, Kenneth Kavajecz, Pete Kyle, Mike Lemmon, Venkatesh Panchapagesan, Gideon Saar, Raj Singh, and an anonymous referee for their helpful comments. An earlier version of this paper was presented at the annual meeting of the American Finance Association, January 1998. Contact the author at finsb@business.utah.edu.

(*Journal of Business,* 2005, vol. 78, no. 4)

0021-9398/2005/7804-0005$10.00

with the limit orders that accumulated in his electronic book. The specialist finds a single price that will clear the market order imbalance as well as all the limit orders to buy (sell) at or below (above) the clearing price. However, unlike a typical auctioneer, the specialist can buy or sell for his own account.[1]

Other exchanges, including the Toronto Stock Exchange, the Paris Bourse, and the Frankfurt Stock Exchange, also begin the trading day with a single-price call-type auction. The conventional wisdom is that a single-price auction is a good way to establish a price that reflects broad interest. With an average of 10% of the daily dollar trading volume on the NYSE taking place at the opening, it seems that many investors prefer trading at the opening.[2] The NYSE also uses the single-price auction after a trading halt, when uncertainty is high, creating the need to establish a single price that aggregates diverse views of investors.[3]

To study the effects the change in the transparency of the limit-order book might have, we employ a stylized model of a specialist's single-price auction in two different environments: in one environment, the limit-order book is open; in the other, it is closed. In our model, liquidity traders submit market orders. These are paired automatically, and the market order imbalance is presented to the specialist. A finite number of strategic off-exchange limit-order traders submit price-contingent orders that are placed in the limit-order book. To study the efficiency of the price discovery process, our model incorporates a strategic informed trader who places a market order. The strategic specialist, after observing the market order imbalance and the limit-order book, sets the price that clears the market order imbalance, with the book taking precedence over the specialist.

Our results show that, when the market is large enough, opening the limit-order book is beneficial to market order traders, whether informed or liquidity. In fact, the price impact of market orders (reciprocal of depth) is lower on average when the book is open, so the cost of trading is lower on average with an open book, implying fewer price reversals after the opening. Moreover, we show that, on average, prices reveal more information when the book is open, implying lower post-open volatility. This result contrasts with the belief that efficiency of prices

1. See Stoll (1985) for an in-depth study of the economics of the specialist's roles in the NYSE.

2. See Madhavan and Panchapagesan (2000) who also report that, for low-volume stocks, the opening can count for as much as 25% of the total daily volume.

3. Empirical studies of single-price auction have been made by Stoll and Whaley (1990), Biais, Hillon, and Spatt (1999), and Madhavan and Panchapagesan (2000). Stoll and Whaley (1990) studied the opening on the NYSE. They found that prices tend to reverse around the opening, and they concluded that the immediacy suppliers do extract rents from the liquidity traders. Biais et al. (1999) studied the opening in the Paris Bourse, which is an open book environment where a disinterested auctioneer, a computer, sets the clearing price. They suggest that the preopening inductive prices converge to an efficient opening price.

comes at the expense of the liquidity traders (see, e.g., O'Hara 1995, p. 271). Our results are driven by the interaction between the two components of trading costs (the adverse selection and the transitory component), which is endogenous. When the book is open, the transitory component is lower, due to the increase in competition for liquidity provision. Thus, the informed trader trades more aggressively, releasing more of his private information. However, the decrease in the transitory component offsets the increase in the adverse selection component, so that overall trading costs are lower and prices are more informative in the open-book environment.

We also show that limit-order traders extract more rents when the book is closed, and numerical analyses indicate that the specialist, too, is better off in the closed-book environment. These results can be explained in the following way. In the closed-book environment, the specialist and the limit-order traders enjoy informational advantages. The specialist observes the complete structure of limit-order book, while the limit-order traders have partial knowledge of the book's structure; namely, each knows that the book contains his order. These advantages do not exist in the open-book environment. Furthermore, our results are robust with respect to the distribution of noise that we introduce into the book, as long as the market is large enough. Thus, our model demonstrates how pretrade transparency allows limit-order traders to compete more effectively with the specialist and consequently to reduce his monopoly rents.

A shortcoming of our model is a restriction we impose on the limit-order traders. Due to the difficulty of solving the limit-order traders' problem, we restrict those traders' strategies to the class of linear demand schedules. To study the practical importance of the restriction, we develop an unrestricted model that focuses solely on the limit-order traders' problem (i.e., no specialist and the informativeness of the order flow is taken as given). We used the unrestricted model to verify that the average outcomes of a restricted model are similar to those of the unrestricted one. Moreover, we show that, for small market orders, the open-book environment provides more liquidity. In contrast, for large market orders, the opposite is true. However, as in the restricted model, on average, the open-book environment is superior in terms of liquidity provision.

Limit-order traders provide better prices for large market orders in a closed-book environment because limit-order traders can condition their orders only on prices. Conditioned on extreme prices, limit-order traders have to consider two possibilities: the extreme price is due to either lack of depth in the book or a large market order. In the former case, limit-order traders extract high rents; while in the latter, they are likely trading against informed traders. Competition among limit-order traders drives their expected profit down, so that, in fact, they lose when they trade against a large market order.

Our paper is closely related to the growing literature on the limit-order book. While the current paper focuses on the limit-order book at the opening, most papers that model the book are interested in the discriminatory price auction that follows the opening. Whereas the trading protocol at the opening is a single-price auction, the continuous trading protocol is a discriminatory price auction. That is, a large market order is paired off with several limit orders, possibly at different prices.[4] Our model contributes to this literature in several ways. Ours is the only model in which strategic limit-order traders, a strategic market order trader (the informed trader), and a strategic specialist interact. In particular, this interaction allows us to study the strategy of a limit-order trader who knows that his actions alter the behavior of the specialist. Moreover, ours is the only study of trading into a closed, random depth limit-order book.

Our model is also related to the literature on transparency. Madhavan (1995) studied the effect of posttrade transparency. In a market without a posttrade disclosure requirement, dealers may be willing to provide better quotes to compete on order flow for its information content. Pagano and Roel (1996) studied different trading systems with different degrees of transparency. However, these systems are different in dimensions other than their degrees of transparency. Bloomfield and O'Hara (1999), and Flood et al. (1999) use experiments to study transparency in a pure dealer markets.

Boehmer, Saar, and Yu (2005) is an empirical study of the issues discussed in our work. They contrast market outcomes before and after the introduction of the OpenBook system and provide evidence to the main predictions made here. More specific, they find that, after the NYSE introduced the OpenBook system, the "ex ante" liquidity (as reflected by the book) and the "ex post" liquidity (taking into account price improvement by the specialist) improved. They also found some improvement in efficiency of prices following the introduction of the OpenBook. Interestingly, Boehmer et al. (2005) also compared the rate of alteration of limit-order traders before and after the introduction of OpenBook, finding that in the transparent environment the rate is higher. This is an indication that indeed the OpenBook facilitates effective competition between the limit-order traders. Each trader observes the book and adjusts his order. On the other hand, Madhavan, Porter and Weaver (1999) studied the 1990 move of the Toronto Stock Exchange into a

4. For example, Rock (1990), Seppi (1997), and Parlour and Seppi (2001) model the limit-order book in a transparent, continuous trading environment with competitive limit-order traders and a strategic specialist. Glosten (1994) is yet another variant of the Rock model; however, without the specialist. Parlour (1998) and Foucault (1999) model dynamic limit-order markets, i.e., without specialist, using the Glosten and Milgron framework. Chakravarty and Holden (1995) model a market in which the informed can submit limit orders; although, in their model, market makers ignore the information in the limit-order book.

transparent environment. They found higher spreads and higher volatility in the transparent environment. Madhavan et al. (1999) express the view that, in a transparent limit-order book environment, an informed trader can better place his market orders. This should result in higher gains for him at the expense of the limit-order traders. So, the argument goes, limit-order traders are reluctant to post their orders for fear of being picked off by an informed trader, resulting in a thinner limit-order book. However, our model shows that another argument can be made: regardless of transparency, one expects to find less-informed trading and less gains for informed traders when the book is thin. Because liquidity providers can better compete in a transparent environment, the limit-order book should be thicker in an open-book environment.

The paper is organized as follows. Section II describes the primitives of the model. Sections III and IV derive the linear restricted closed book equilibrium and the linear restricted open book equilibrium, respectively. Section V compares the equilibria. Section VI develops the unrestricted model. Finally, Section VII concludes the paper.

II. The Model

We consider a call market for a risky asset and a risk-free asset (numeraire) with the interest rate set to zero. At time 1, the risky asset pays $\tilde{v}$. We study an equilibrium in two different environments. In one environment, the limit order book is open, while in the other only the specialist observes the book. The characteristics of both environments are presented next, followed by a discussion.

Four types of participants are in our market. The first group consists of the liquidity traders. We do not model their behavior.[5] We denote their aggregate market orders by $\tilde{z}$. One strategic risk-neutral informed trader, who knows the realization of $\tilde{v}$, submits a market order, $\tilde{x}$.[6] The aggregate market order, $\tilde{y}$, which we sometimes call the *market order imbalance,* is equal to $\tilde{z} + \tilde{x}$.

Demand schedules are submitted to the specialist by $\tilde{N}$ strategic risk-neutral traders, who are called *limit-order traders.*[7] For the distinction

5. One could model the demand of liquidity trading. We have chosen not to do this for two reasons. First, linear equilibria fail to exist unless liquidity traders are risk averse. Risk aversion strengthens the bias in favor of an open-book environment. Second, we expect endogenous liquidity demand to sharpen our results. This is because now, with elastic liquidity demand, liquidity demands are greater in the environment with the lower cost of trading. Furthermore, one well-known stylized fact in finance, which finds ground in the asymmetric information literature too, is that higher volume reduces cost of trading (see Demsetz 1968).

6. In our risk-neutral environment, one can assume that $\tilde{v}$ is merely an unbiased estimator of the liquidation value.

7. A limit order is a single-step function. It sets the upper (lower) price at which a trader is willing to buy (sell) up to a specified quantity. It seems reasonable that, with decimal pricing, limit-order traders submit multiple limit orders, thus mimicking a demand schedule (see Kyle 1989).

between open and closed books to be meaningful, there must be some uncertainty about the book. In this paper, noise is introduced into the book through the number of limit-order traders, who are assumed to be drawn out of a pool of potential limit-order traders. Furthermore, conditional on the realization of $\tilde{N}$, each potential limit-order trader is equally likely to be present in the market. We denote by $f_i(\cdot)$ the ith limit-order trader's demand schedule with the interpretation that, at price p, $f_i(p)$ is the quantity the trader demands. It is convenient to denote the book's randomness by writing $f(\tilde{N}, \cdot) = \sum_{i=1}^{\tilde{N}} f_i(\cdot)$. We study equilibria in which all the limit-order traders make the same choice of a demand schedule.

The role of the specialist is to set a single price and clear the market. As a dealer, the specialist can buy and sell for his own account. However, he is subject to one important restriction. At the clearing price, the first transactions go to the book. This restriction prevents the specialist from setting an arbitrarily high (low) price and selling (buying) all the excess market orders.

Given the price, p, chosen by the specialist, the informed trader's profit is

$$(\tilde{v} - p)x, \tag{1}$$

and the profit of the ith limit-order trader is

$$(\tilde{v} - p) f_i(p). \tag{2}$$

The specialist receives the quantity

$$-\tilde{x} - \tilde{z} - \sum_{i=1}^{\tilde{N}} f_i(p),$$

so his profit is

$$(p - \tilde{v})\left(\tilde{x} + \tilde{z} + \sum_{i=1}^{\tilde{N}} f_i(p)\right). \tag{3}$$

Our probability space has three independent random variables: $\tilde{v}$, $\tilde{z}$, and $\tilde{N}$. The liquidation value $\tilde{v}$ is normally distributed with mean $\bar{v}$ and variance σ_v^2. The aggregate liquidity order $\tilde{z}$ is normally distributed with mean zero and variance σ_z^2. The numer of limit-order traders, $\tilde{N}$, is a bounded positive integer-valued random variable. A lower bound on the support of $\tilde{N}$ is needed for certain results. We impose no other distributional assumptions on $\tilde{N}$. Given a random variable $\tilde{u}$, the notations u and $\bar{u}$ are used to denote its realization and its expected value, respectively.

Due to the mathematical difficulty of solving the limit-order traders' problem, we can use only approximation methods. Our approach is to solve analytically an approximate model: we restrict the limit-order traders to linear demand schedules.[8]

Our model does not incorporate the group of floor brokers because we want a level playing field. The floor traders provide an advantage for their clients whether the book is closed or open. In a closed-book environment, they can communicate information from the floor to their off-exchange clients, in particular, tell them what is in the limit-order book. On the other hand, in an open limit-order book environment they can "work" the orders of their clients rather than posting limit orders. Interestingly, Sofianos and Werner (1997) found that the participation of floor brokers at the opening is very low. They estimated the value of floor brokers' executed orders at the opening, excluding orders submitted through the OARS, to be only 0.9%.[9]

III. Closed Book

In the closed-book environment, the informed trader can condition his market order only on the asset value $\tilde{v}$. He has no information about the book when he submits his order. We therefore write his market order as a function $x(v)$.

An important feature of a closed-book environment is that a limit-order trader knows his own demand schedule, f_i, and thus possesses some information on the book's content. Here, this information is captured by the fact that a limit-order trader knows that he is active in the market; that is, the trader knows that the book contains his order. Let m_i be the indicator function of the event that the ith trader is active; that is, $m_i = 1$ when he is active and $m_i = 0$ otherwise.

The specialist observes both the market order imbalance

$$\tilde{y} \equiv x(\tilde{v}) + \tilde{z}$$

and the book $f(\tilde{N}, \cdot)$ before choosing the price p, so he chooses the price as a function of $\tilde{y}$ and $f(\tilde{N}, \cdot)$. We write this function as $P(y, f)$.

8. Using the unrestricted model presented in Section VI, we can show that the smaller the information content in order flow, the smaller the expected price deviation a limit-order trader expects and, hence, the better the linear approximation is. This result is available from the author on request. We also show (see corollary 3) that, when the number of limit-order traders is very large, our equilibrium outcomes approach the outcomes of the competitive and unrestricted model, and we obtain this convergence result even for large expected price deviations.

9. Also Boehmer et al. (2005) found that, after OpenBook was introduced, volume attributed to floor brokers declined relative to volume attributed to the limit-order book.

The informed trader's expected profit from a market order x, contingent on a realization v of the random variable $\tilde{v}$, is

$$E\{v - P[x + \tilde{z}, f(\tilde{N}, \cdot)]x\}. \tag{4}$$

The ith limit-order trader takes the demand schedules of the other limit-order traders as given. It is convenient to focus on the decision problem of the first trader, since all the limit-order traders face the same decision problem. Given f_i for $j > 1$, set

$$f_{-1}(\tilde{N}, p) = \sum_{j=2}^{\tilde{N}} f_j(p).$$

Given a demand schedule f_1, the book is

$$f_1(\cdot) + f_{-1}(\tilde{N}, \cdot).$$

The expected profit of the limit-order trader, contingent on the knowledge that he is active in the market, is

$$E[(\tilde{v} - \tilde{p})f_1(\tilde{p})|m_i = 1], \quad \text{where } \tilde{p} \equiv P[\tilde{y}, f_1(\cdot) + f_{-1}(\tilde{N}, \cdot)] \tag{5}$$

Given a market order y and a book $f(\tilde{N}, \cdot)$, the specialist chooses the price p to maximize

$$E\{(p - \tilde{v})[y + f(\tilde{N}, p)]|f(\tilde{N}, \cdot), x(\tilde{v}) + \tilde{z} = y\}. \tag{6}$$

A linear restricted equilibrium consists of a decision rule $x(v)$ for the informed trader, a demand schedule f_1 for each of the limit-order traders, and a decision rule $P(y, f)$ for the specialist such that

1. The market order $x(v)$ maximizes (4) for each realization v of $\tilde{v}$.
2. The demand schedule f_1 maximizes (5) over the class of linear functions.
3. The price rule $P(y, f)$ maximizes (6) for each realization y of $x(\tilde{v}) + \tilde{z}$ and each linear demand schedule f.

Theorem 1. If $E\left[\frac{\tilde{N}-2}{\tilde{N}^3}\middle| m_i = 1\right] > 0$, then there exists a linear restricted equilibrium (hereafter, equilibrium) in which[10]

(i) The ith limit-order trader's demand schedule has the form

$$f_i(p) = (\tilde{v} - p)B_c.$$

10. We use the subscript c to indicate the closed-book environment.

(ii) The informed trader's decision rule has the form

$$x(\tilde{v}) = \beta_c(v - \bar{v}),$$

(iii) The specialist's price rule has the form

$$P(y, f) = \bar{v} + \frac{1}{2}\left[b_c + \frac{1}{-f'(p)}\right]y + \frac{1}{2}\frac{f(\bar{v})}{-f'(p)}$$

In particular, since $f'(p) = NB_c$ and $f(\bar{v}) = 0$, the pricing rule in equilibrium simplifies to $p = B - V + \tilde{\lambda}_c y$ where $\tilde{\lambda}_c \equiv \frac{1}{2}\left(b_c + 1/\tilde{N}B_c\right)$.

The triple (B_c, b_c, β_c) is given as the positive solution of the following system of equations:

$$\begin{cases} b_c = \frac{\beta_c \sigma_v^2}{\beta_c^2 \sigma_v^2 + \sigma_z^2} \\ \tilde{\lambda}_c = \frac{1}{2}\left(b_c + \frac{1}{B_c}\frac{1}{\tilde{N}}\right) \\ B_c = \frac{1}{b_c}\sqrt{E\left[\frac{\tilde{N}-2}{\tilde{N}^3}\middle| m_i = 1\right]} \\ \beta_c = \frac{1}{2E\tilde{\lambda}_c}. \end{cases} \tag{7}$$

Proof. Before we show that the system (7) defines an equilibrium, we need to show that the system possesses a solution. If a solution exists, then it implies that

$$\begin{cases} b_c = \frac{\beta_c \sigma_v^2}{\beta_c^2 \sigma_v^2 + \sigma_z^2} \\ E\tilde{\lambda}_c = \frac{1}{2}\left(b_c + \frac{1}{B_c}E\frac{1}{N}\right) \\ B_c = \frac{1}{b_c}\sqrt{E\left[\frac{\tilde{N}-2}{\tilde{N}^3}\middle| m_i = 1\right]} \\ \beta_c = \frac{1}{2E\tilde{\lambda}_c}. \end{cases} \tag{8}$$

It is straightforward to see that a unique positive solution to this system exists. Endowed with $E\tilde{\lambda}_c$, we can solve the system (7), where the primitives are σ_v, σ_z, $E\left[\frac{\tilde{N}-2}{\tilde{N}^3}\middle| m_i = 1\right]$, $E\tilde{\lambda}_c$, and the realization N of $\tilde{N}$.

The proof that the system (7) defines a linear restricted equilibrium is given in Appendix A. Q.E.D.

The assumption that $\tilde{N} \geq 2$ and is nondegenerate is sufficient for the existence of an equilibrium. It is, however, not necessary. What is important is that a limit-order trader does not assign too much weight to the event that he has monopoly power, that is, the event $\{\tilde{N} = 1\}$.

The equilibrium we found has several features that distinguish it from what has been done so far in the literature. Here, not only does a strategic limit-order trader utilize information in the clearing price by conditioning his demand on the opening price, he also takes into account the strategy of the specialist who chooses his position only after all the orders have been submitted to him.[11] Furthermore, because the number of traders in our model is uncertain, the price impact of a market order, measured by $\tilde{\lambda}_c$, is random. Neither a limit-order trader nor the informed trader observes $\tilde{\lambda}_c$, although, as we mentioned, a limit-order trader possesses some information about it. It turns out, as the following lemma demonstrates, that there is a simple way to express the statistical value of that information which we denote by m_i.

LEMMA 1. The ratio of conditional to unconditional probabilities of $\tilde{N}$ is

$$\frac{\text{Prob}(\tilde{N}=N|m_i=1)}{\text{Prob}(\tilde{N}=N)}=\frac{N}{E\tilde{N}}.$$

In particular, for any $g(\cdot)$,

$$E\left[g(\tilde{N})|m_i=1\right]=\frac{Eg(\tilde{N})\tilde{N}}{E\tilde{N}}.$$

Proof. Since each potential limit-order trader is chosen with the same probability out of the pool of potential limit-order traders, we have $\text{Prob}(m_i=1|\tilde{N}=N)=N/K$, where K is the number of potential traders.[12] This implies that

$$\begin{aligned}\text{Prob}(m_i=1)&=\sum_N \text{Prob}(m_i=1,\tilde{N}=N)\\&=\sum_N \text{Prob}(m_i=1|\tilde{N}=N)\text{Prob}(\tilde{N}=N)\\&=\frac{E\tilde{N}}{K}\end{aligned}$$

and hence

$$\begin{aligned}\text{Prob}(\tilde{N}=N|m_i=1)&=\text{Prob}(\tilde{N}=N)\frac{\text{Prob}(m_i=1|\tilde{N}=N)}{\text{Prob}(m_i=1)}\\&=\text{Prob}(\tilde{N}=N)\frac{N}{E\tilde{N}}.\end{aligned}$$

Q.E.D.

11. Other models that model a strategic specialist, such as Rock (1990), assume the limit-order traders are nonstrategic.
12. K is the upper bound on the support of the distribution of $\tilde{N}$, which we assume to exist.

Intuitively, we expect that the larger is the number of limit-order traders, the less valuable the information a limit-order trader has. Indeed, the lemma shows that the larger the values $\tilde{N}$ can take, the closer to 1 is the ratio of conditional to unconditional probabilities. However, the equilibrium outcomes are determined by aggregation. Thus, even with a large expected number of limit-order traders, we cannot rule out the informational advantage limit-order traders possess in a closed-book environment. The conditional expectation that appears in (8), $E[(\tilde{N}-2)/\tilde{N}^3|m_i=1]$, is equal to $E(\tilde{N}-2)/\tilde{N}^2 1/EN$. It is convenient to rewrite the system of equations (8) as

$$\begin{cases} b_c &= \frac{\beta_c\sigma_v^2}{\beta_c^2\sigma_v^2+\sigma_z^2} \\ E\lambda_c &= \frac{1}{2}\left(b_c+\frac{1}{B_c}E\frac{1}{\tilde{N}}\right) \\ B_c &= \frac{1}{b_c}\sqrt{E\frac{\tilde{N}-2}{\tilde{N}^2}\frac{1}{E\tilde{N}}} \\ \beta_c &= \frac{1}{2E\lambda_c}. \end{cases} \tag{9}$$

Lemma 1 helps us gain some insight into the equilibrium in the closed-book environment. The lemma implies that, whenever N is greater than $E\tilde{N}$, the conditional probability of $\tilde{N}$ with respect to the event $\{m_i=1\}$ assigns more weight to the event $\{\tilde{N}=N\}$ than does the unconditional probability. It follows that each of the limit-order traders expects the price impact of a market order to be smaller than its unconditional average. Indeed, from the second equation in (9),

$$E[\tilde{\lambda}_c\,|\,m_i=1]=\frac{1}{2}\left[b_c+\frac{1}{B_c}E\left(\frac{1}{N}\middle|m_i=1\right)\right]=\frac{1}{2}\left(b_c+\frac{1}{B_c}\frac{1}{EN}\right)\leq E\tilde{\lambda}_c,$$

where the second equality follows from lemma 1 with $g(N)=1/N$.

The liquidity that a limit-order trader provides is inversely related to his belief about the aggregate liquidity provided by the market. One could argue that, since limit-order traders overestimate aggregate liquidity (i.e., underestimate $\tilde{\lambda}_c$), opening the book should increase liquidity provision. To make this statement precise, we need first to know how the specialist and the informed trader will revise their strategies in response to opening the book. This is the aim of the next section.

IV. Open Book

In this section, we would like to remove some of the specialist's informational advantage by opening the book. To be consistent with the NYSE OpenBook specifications, the specialist does not disclose the market-order imbalance. According the the NYSE, "In some cases, market orders comprise the majority of pre-opening interest, and market order imbalances become the key determinant to where a stock will open."[13] Thus, the book alone cannot indicate the opening price.[14]

Modeling the dynamic of an open-book environment is a complicated task. Instead, the approach taken in this paper is to assume that when the market is called the book is in a state of equilibrium; that is, given the book's status, no single limit-order trader desires to change his order. We continue, as in the closed-book environment, to maintain the role of the specialist as the "follower," who takes his actions only after the book has reached equilibrium. This time, however, the specialist has no informational advantage, since everyone sees the book before the market is called.

A book in a state of equilibrium is the one that results from a static Bayesian Nash equilibrium in pure strategies under the assumption that N is common knowledge. In such an equilibrium, each of the traders perfectly predicts the book's structure before submitting his order. In particular, once the book is realized, no trader desires to change his order, and the specialist can call the market, that is, announce the price and clear the market.

Therefore, we consider an equilibrium where the informed trader's market order is a function $x(v, N)$, a demand schedule is a linear function $f_1(N, \cdot)$, and the price rule is $P(N, y, f)$. The informed trader's expected profit from a market order x is

$$E\{v - P[N, x + \tilde{z}, f(N, \cdot)]x\}. \tag{10}$$

The expected profit of the limit-order trader is

$$E(\tilde{v} - \tilde{p}) f_1(\tilde{p}), \quad \text{where } \tilde{p} \equiv P[N, \tilde{y}, f_1(\cdot) + f_{-1}(N, \cdot)]. \tag{11}$$

Given a market order y and a book f, the specialist expected profit is

$$E\{(p - \tilde{v})[y + f(p)] | x(\tilde{v}, N) + \tilde{z} = y\}. \tag{12}$$

A linear restricted equilibrium consists of a decision rule $x(v, N)$ for the informed trader, a decision rule $f_1(N, \cdot)$ for each of the limit-order traders, and a decision rule $P(N, y, f)$ for the specialist such that

1. The market order $x(v, N)$ maximizes (10) for each realization v of $\tilde{v}$.
2. The demand schedule $f_1(N, \cdot)$ maximizes (11) over the class of linear functions.
3. The price rule $P(N, y, f)$ maximizes (12) for each realization y of $x(N, \tilde{v}) + \tilde{z}$ and each linear demand schedule f.

13. See www.nysedata.com/openbook/FAQ.htm.

14. This is in contrast with the Paris Bourse, where each time a new order is placed, a new inductive price is announced.

THEOREM 2. If $N > 2$, then there exists a linear restricted equilibrium (hereafter, equilibrium) in which[15]

(i) The demand schedule is given by

$$f_1(N, p) = (\bar{v} - p)B_o(N).$$

(ii) The informed-trader decision rule is given by

$$x(\tilde{v}, f) = \beta_o(N)(\tilde{v} - \bar{v}).$$

(iii) The price rule has the form

$$P(N,y,f) = \bar{v} + \frac{1}{2}\left[b_o(N) + \frac{1}{-f'(p)}\right]y + \frac{1}{2}\frac{f(\bar{v})}{-f'(p)}.$$

In particular, in equilibrium, $p = \bar{v} + \lambda_o(N)y$, where

$$\lambda_o(N) \equiv \frac{1}{2}\left[b_o(N) + \frac{1}{NB_o(N)}\right].$$

The triple $(B_o(N), \beta_o(N), b_o(N))$ is the positive solution of the following system of equations:

$$\begin{cases} b_o = \frac{\beta_c \sigma_v^2}{\beta_o^2 \sigma_v^2 + \sigma_z^2} \\ \lambda_o = \frac{1}{2}\left(b_o + \frac{1}{NB_o}\right) \\ B_o = \frac{1}{b_o}\sqrt{\frac{N-2}{N^3}} \\ \beta_o = \frac{1}{2\lambda_o}. \end{cases} \tag{13}$$

Proof. This is merely a special case of theorem 1, in which the distribution that governs $\tilde{N}$ is degenerate. Indeed, once we consider $\tilde{N}$ as known, system (7) reduces to (13). Q.E.D.

Despite the pretrade transparency, the semi-strong-efficient condition, $\tilde{p} = E[\tilde{v} \mid \tilde{p}]$, does not hold in equilibrium because of the market power of the liquidity providers. In fact, the specialist's and the value traders' expected gains are strictly positive. However, we can prove the following.

COROLLARY 3. In the limit, as the lower bound of N goes to infinity, the equilibrium in the open-book environment converges to the one found in Kyle (1985). In particular, in the limit the specialist acts as an auctioneer.

Proof. The market efficiency condition holds if and only if $\lambda_o = b_o$ (see Kyle 1985). From the second equation in (13), this condition holds

15. We use the subscript o to indicate the open-book environment.

if the competition among the value traders results in $1/NB_o = b_o$. It follows from the specialist price rule that, in that case, the specialist takes no position. From the third equation in (13), it follows that

$$\frac{1}{NB_o} = \sqrt{\frac{N}{N-2}}\, b_o > b_o.$$

However, as N goes to infinity, $\sqrt{N/(N-2)}$ goes to 1 and prices become efficient. Q.E.D.

V. Comparison of Equilibria

Due to the risk-neutrality assumption, the model we presented is a zero-sum game. Hence, moving from one environment to the other cannot benefit everyone. In this section, we determine who gains from the closed-book environment and who gains from moving to the open-book environment.

It is convenient to introduce the change of variables,

$$\begin{aligned} \tilde{a} &:= \frac{1}{\tilde{N}} \\ \tilde{r} &:= \tilde{a} - 2\tilde{a}^2 = \frac{\tilde{N}-2}{\tilde{N}^2}, \end{aligned} \tag{14}$$

and express the solution of the closed book equilibrium (system [9]) in terms of $E\tilde{N}$, $E\tilde{a}$, and $E\tilde{r}$:[16]

$$\begin{cases} b_c &= b(E\tilde{N}, E\tilde{a}, E\tilde{r}) \\ \beta_c &= \beta(E\tilde{N}, E\tilde{a}, E\tilde{r}) \\ E\tilde{\lambda}_c &= \lambda(E\tilde{N}, E\tilde{a}, E\tilde{r}) \\ B_c &= B(E\tilde{N}, E\tilde{a}, E\tilde{r}). \end{cases}$$

We note that the same functional form of the right-hand side can be used to express the realization of the open book equilibrium, that is, system (13):

$$\begin{cases} \tilde{b}_o = b(\tilde{N}, \tilde{a}, \tilde{r}) \\ \tilde{\beta}_o = \beta(\tilde{N}, \tilde{a}, \tilde{r}) \\ \tilde{\lambda}_o = \lambda(\tilde{N}, \tilde{a}, \tilde{r}) \\ \tilde{B}_o = B(\tilde{N}, \tilde{a}, \tilde{r}). \end{cases}$$

16. There is no ambiguity regarding σ_v and σ_z. Hence, we treat them as parameters and omit them.

The functional form is given by

$$\begin{cases} b(N,a,r) = \frac{\sigma_v}{\sigma_z}\frac{\sqrt{a\sqrt{\frac{r}{N}}}}{a+\sqrt{\frac{r}{N}}} \\ \beta(N,a,r) = \frac{\sigma_z}{\sigma_v}\sqrt{\frac{\sqrt{\frac{r}{N}}}{a}} \\ \lambda(N,a,r) = \frac{1}{2}\frac{\sigma_v}{\sigma_z}\sqrt{\frac{a}{\sqrt{\frac{r}{N}}}} \\ B(N,a,r) = \frac{\sigma_z}{\sigma_v}\sqrt{\frac{\sqrt{\frac{r}{N}}}{a}}\left(a+\sqrt{\frac{r}{N}}\right). \end{cases} \tag{15}$$

If λ (N, a, r) were a concave or convex function, we would use Jensen's inequality to determine under which environment the equilibrium expected price impact is smaller. Unfortunately, this is not the case. Nevertheless, since *a, r,* and *N* are related, we can come up with a definite answer.

LEMMA 2. If the support of $\tilde{N}$ has a lower bound greater than 8, then

(i) The expected equilibrium price impact of a market order in an open-book environment is smaller than in a closed-book environment.
(ii) The informed trader's intensity of trade in the closed-book equilibrium is smaller than his expected intensity of trade in the open-book equilibrium.

Proof. We need to show that $E\lambda(\tilde{N}, \tilde{a}, \tilde{r}) \leq \lambda(E\tilde{N}, E\tilde{a}, E\tilde{r})$, which is equivalent to the relation

$$E\left((\tilde{N})^{\frac{1}{4}}\sqrt{\frac{\tilde{a}}{\sqrt{\tilde{r}}}}\right) \leq \sqrt{\sqrt{E\tilde{N}}}\sqrt{\frac{E\tilde{a}}{\sqrt{Er}}}.$$

The proof of the latter is given by

$$E\left((\tilde{N})^{\frac{1}{4}}\sqrt{\frac{\tilde{a}}{\sqrt{\tilde{r}}}}\right) \leq \sqrt{E\sqrt{\tilde{N}}}\sqrt{E\frac{\tilde{a}}{\sqrt{\tilde{r}}}} \leq \sqrt{\sqrt{E\tilde{N}}}\sqrt{\frac{E\tilde{a}}{\sqrt{r(E\tilde{a})}}} \leq \sqrt{\sqrt{E\tilde{N}}}\sqrt{\frac{E\tilde{a}}{\sqrt{Er}}},$$

where the first inequality is Cauchy-Schwartz, the second inequality follows from the fact that the function $a \to a/\sqrt{r} \equiv a/\sqrt{a-2a^2}$ is concave on the interval $[0, \frac{1}{8}]$, and the third inequality follows from the concavity of $r(a)$.

The second part of the lemma follows immediately from the relation $\beta(N, a, r) = 1/[2\lambda(N, a, r)]$ and Jensen's inequality. Q.E.D.

The condition $N \geq 8$ means that our results are definitive as long as the market is large enough. Since our aim is to model a market similar to the NYSE, the largest stock exchange in the world, we do not view the condition as a real limitation. Moreover, the condition is only sufficient to ensure these results. In Appendix B, we give examples in which the results of lemma 2 hold, even in the case that the whole support of N lies below 8.

Given the results in lemma 2, we are ready to prove our main theorem.

THEOREM 4. If the support of $\tilde{N}$ has a lower bound greater than 8, then

(i) The expected losses that the liquidity traders incur in an open-book equilibrium are smaller than those in a closed-book equilibrium.
(ii) The informed trader's expected profit is higher in an open-book equilibrium than in a closed-book equilibrium.

Proof. The liquidity traders' aggregate expected losses are $E(\tilde{v} - \tilde{p})\tilde{z}$. Using the independence of $\tilde{v}$, $\tilde{z}$, and $\tilde{N}$ we have, in both equilibria,

$$\begin{aligned} E(\tilde{v} - \tilde{p})\tilde{z} &= E\{\tilde{v} - v - \tilde{\lambda}[\tilde{z} + \tilde{\beta}(\tilde{v} - v)]\}\tilde{z} \\ &= -\sigma_z^2 E\tilde{\lambda}, \end{aligned}$$

and the first part of the theorem follows from lemma 2.

In both equilibria, the informed trader's expected profit is

$$\begin{aligned} E(\tilde{v} - \tilde{p})\beta(\tilde{v} - \tilde{v}) &= E\{\tilde{v} - \tilde{v} - \lambda[\tilde{z} + \beta(\tilde{v} - \bar{v})]\}\beta(\tilde{v} - \bar{v}) \\ &= \frac{1}{2}\sigma_v^2 E\beta \end{aligned}$$

where the last equality follows from the zero expectation of $\tilde{z}$; the independence of $\tilde{z}$, $\tilde{v}$, and $\tilde{N}$ (and hence λ, which is only a function of $\tilde{N}$); and the relation $E\lambda\beta^2 = \frac{1}{2}E\beta$, which holds in both equilibria (see the fourth equation in [7] and the fourth one in [13]).

We conclude that, to compare the ex ante profit, we should compare the expected value of the intensity of trade. Hence, the result follows from the second part of lemma 2. Q.E.D.

Since, on average, the cost of immediacy is higher and the informed trader's intensity of trade is smaller in the closed-book environment than in the open-book environment, the following result readily follows.

THEOREM 5. If the support of $\tilde{N}$ has a lower bound greater than 8, then the limit-order traders' expected profits in the open-book equilibrium are smaller than in the closed-book equilibrium.

Proof. The following is an outline of the proof; details are given in Appendix B. The limit-order traders' equilibrium expected profits in the closed- and open-book environments can be written as

$$E\tilde{N}(\tilde{v} - \tilde{p}_c)(\bar{v} - \tilde{p}_c)B_c = g(E\tilde{N}, E\tilde{a}, E\tilde{r}),$$
$$E\tilde{N}(\tilde{v} - \tilde{p}_o)(\bar{v} - \tilde{p}_o)B_o = Eg(\tilde{N}, \tilde{a}, \tilde{r}),$$

respectively, where the function g is given by

$$g(N, a, r) = \frac{1}{4}\frac{\sigma_z \sigma_v N^{1/4}(a - r)}{r^{1/4}\sqrt{a}}. \tag{16}$$

Using a series of Jensen's inequalities one can show that

$$Eg(\tilde{N}, \tilde{a}, \tilde{r}) \leq g(E\tilde{N}, E\tilde{a}, E\tilde{r}).$$

Q.E.D.

Last, we consider the specialist. In both environments his expected profit is given by

$$EE[(\tilde{p} - \tilde{v})(\tilde{y} + \tilde{N}B(\bar{v} - \tilde{p}))|\tilde{N}, \tilde{y}] = E(\tilde{p} - \bar{v} - b\tilde{y})(\tilde{y} + \tilde{N}B(\bar{v} - \tilde{p}).$$

Inserting the equilibrium clearing price yields

$$E\frac{1}{4}\tilde{y}^2\frac{(\tilde{N}Bb - 1)^2}{\tilde{N}B}. \tag{17}$$

We define the function $f(N, a, r)$ via

$$f(N, a, r) \equiv \frac{1}{4}[NB(N, a, r)\beta^2(N, a, r)\sigma_v^2 + \sigma_z^2]$$
$$\times [b^2(N, a, r) - 2b(N, a, r) + a/B(N, a, r)].$$

Then, in the closed-book environment, the expected profit, (17), is equal to $f(E\tilde{N}, E\tilde{a}, E\tilde{r})$, while in the open-book environment it is equal to $Ef(\tilde{N}, \tilde{a}, \tilde{r})$. For different distributions of $\tilde{N}$, we compared the two terms numerically and found that the expected profit is higher in the closed-book environment. For example, consider a family of truncated binomial distributions parametrized by p:

$$\text{Prob}(N = i) = \frac{\binom{20}{i} p^i (1 - p)^{20-i}}{\sum_{j=3}^{20} \binom{20}{j} p^j (1 - p)^{20-j}}, \quad i = 3, \ldots, 20. \tag{18}$$

Note that, if the probability p equals 0 or 1, the distribution is degenerate and hence the two equilibria are identical. Under the assumption that

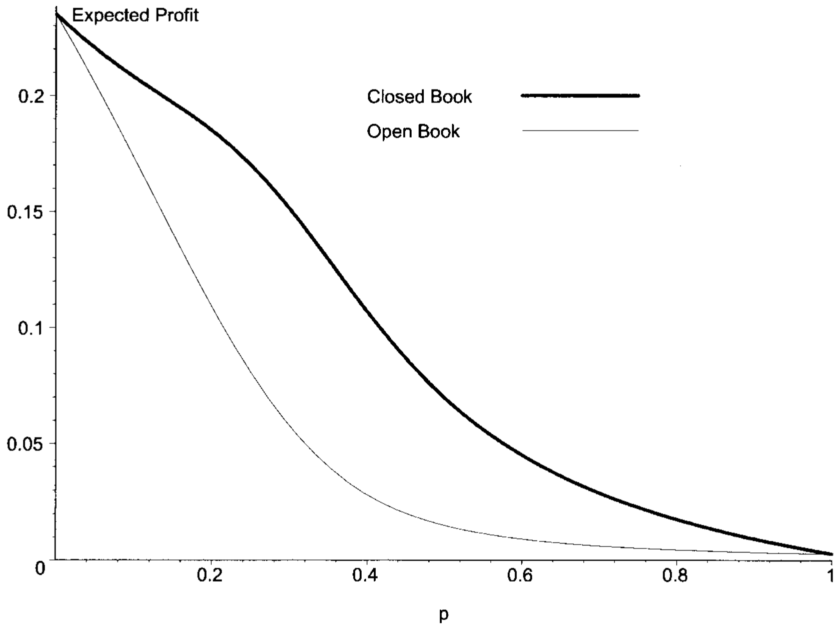

FIG. 1.—The specialist's expected profit under different distributions

$\sigma_z = \sigma_v = 2$, the specialist's expected profit was calculated for different values of p. The results are shown in figure 1.

In lemma 2, we prove that the price impact of a market order is smaller, on average, when the book is open. We can decompose the price impact of market order into its adverse selection component and transitory component, then study the effect transparency has on each component. The transitory component reflects costs induced by the liquidity providers, while the adverse selection component reflects the cost induced by the informed traders. We argue that b_c and b_o are the adverse selection components in a closed- and open-book environment, respectively. Indeed, had the price impact of a market order been equal to the adverse selection component, the liquidity providers' expected profit would have been zero. We showed that, in a closed-book environment, the adverse selection problem is less severe and liquidity providers extract more economic rents. Thus, we expect the adverse selection component to be smaller and the transitory component to be larger in a closed-book environment. For different distributions of $\tilde{N}$, we calculated the average magnitude of each component, and the results verify our intuition. In figures 2 and 3, we show the results for the family of truncated binomial distributions described in (18).

Our next assertion is that, on average, the opening price is more informative in the open-book environment, where $E\ \mathrm{var}(\tilde{v}|\tilde{p})$ is our measure of efficiency. The only source of information in our model is the

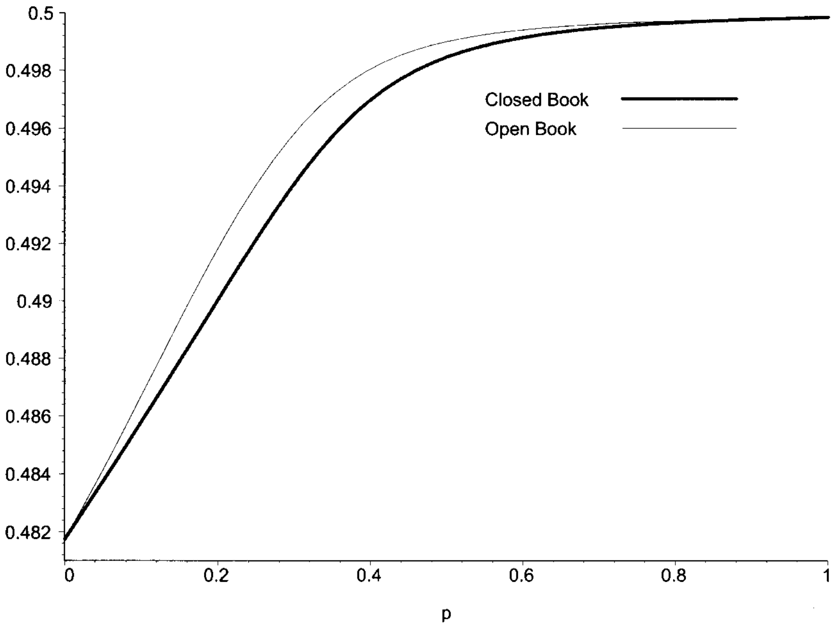

FIG. 2.—The expected adverse selection component under different distributions

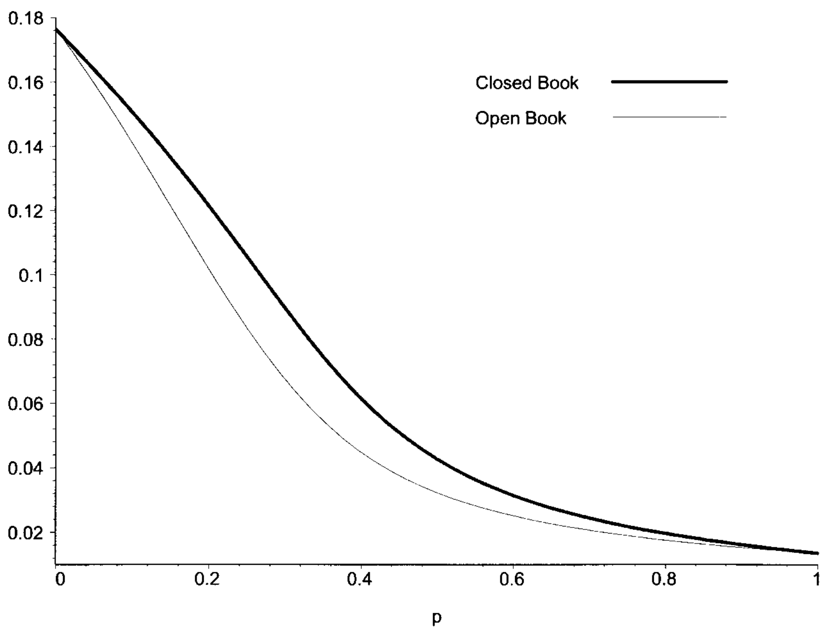

FIG. 3.—The expected transintory component under different distributions

market-order imbalance (the book does not contain private information). In the open-book environment, the traders can infer the market-order imbalance from the opening price; while in the closed-book, the opening price is a noisy signal about the order imbalance due to the randomness in N. Since, in practice, after the opening, traders can learn what the order imbalance was, it is also useful to compare $E\ \mathrm{var}(\tilde{v}|\tilde{y})$.

COROLLARY 6. If the support of $\tilde{N}$ has a lower bound greater than 8, then the conditional variances $\mathrm{var}(\tilde{v}|\tilde{p})$ and $\mathrm{var}(\tilde{v}|\tilde{y})$ are on average smaller in the open-book environment.

The proof of the corollary is given in Appendix D. The result is a direct consequence of the fact that, on average, the informed trader trades more aggressively in the open-book environment (see lemma 2).

VI. Unrestricted Equilibrium Model

The analysis we carried out thus far was tractable due to the restriction of the limit-order trader to the set of linear demand schedules. The purpose of this section is to study the relevance of the restriction. To do so, we consider a market without a specialist, and we take as given the information content in the market-order imbalance. Thus, we focus solely on the strategic behavior of the limit-order traders. Fully consistent with the results in the linear restricted model, we find, on average, limit-order traders are better off when the book is closed and the market is more liquid and less volatile when the book is open.

Without loss of generality, we assume that v and y are symmetric random variables around zero, and we denote by $g(\cdot)$ the density of y.[17] Furthermore, we assume that the order flow is informative; that is, there is an increasing function $b(\cdot)$ such that

$$b(y) = E[\tilde{v}|\tilde{y} = y].$$

As before, there are $\tilde{N}$ strategic limit-order traders. We denote by f_i the demand schedule of the ith strategic limit-order trader and let $f_{-i} = \sum_{j\neq i} f_j$. The clearing price is set by a disinterested auctioneer (computer). Thus, the clearing price satisfies

$$\sum_i f_i(p) + y = 0. \tag{19}$$

We first study the open book environment. As in Section IV, we assume that, when the market is called, no single limit-order trader desires to change his order. We therefore study a static Bayesian Nash equilibrium in pure strategies under the assumption that N is common knowledge.

17. Without the symmetry assumption, we would have to study the buy and sell sides separately.

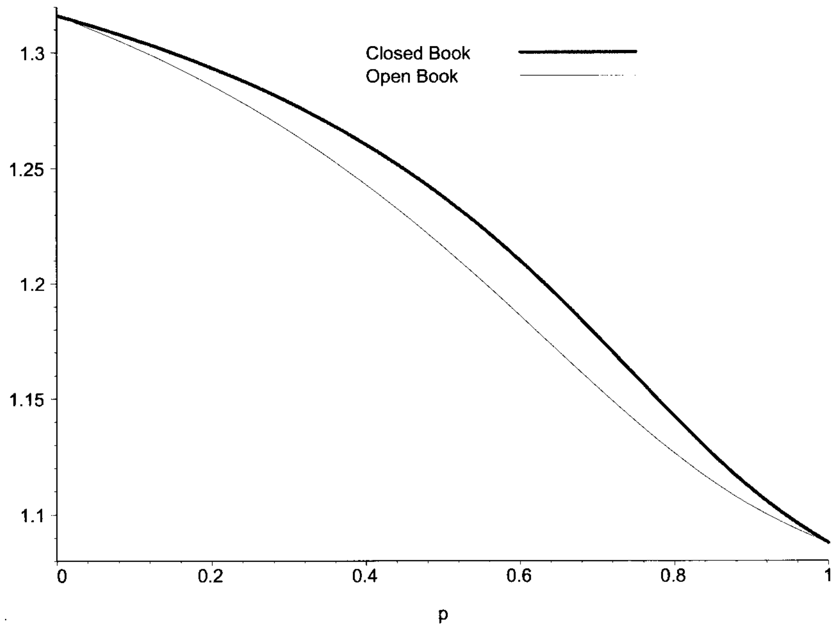

FIG. 4.—Average price impact of market orders for different distributions

The expected profit of a limit-order trader is

$$E(\tilde{v}-\tilde{p})f_i(\tilde{p}), \quad \text{where } f_i(\tilde{p})+f_{-i}(N,\tilde{p})+\tilde{y}=0 \tag{20}$$

An equilibrium consists of demand schedules, f_i, for each of the limit-order traders such that, for each i, f_i maximizes (20) over the class of continuously differentiable functions. A symmetric equilibrium is an equilibrium in which all limit-order traders submit the same demand schedule f.

THEOREM 7. Let $b(y)$ be strictly increasing and twice continuously differentiable. Let f be the solution of the ordinary differential equation (o.d.e.).

$$\frac{f(p)}{f'(p)}=(N-1)\{p-b[-Nf(p)]\}; f(U)=-\infty, f(-U)=\infty, \tag{21}$$

where U is the upper bound of the support of $\tilde{v}$. Then, f defines a symmetric equilibrium in the open-book environment.

The proof of the theorem is given in Appendix E. In equilibrium, the realized limit-order book is simply $Nf(p)$, where f is the solution of the o.d.e. (21). Figure 4 demonstrates how effectively limit-order traders compete away their profits in the open book environment. The figure contrasts the sell side of the limit-order book with the competitive case (i.e., infinite number of limit-order traders). We can see that the realized book is hardly sensitive to the realized number of traders. This is because each limit-order trader adjusts his order in response to what he sees in the book. Therefore, even with relatively few limit-order traders, the equilibrium outcomes are similar to the competitive case.

From the market clearing condition, we know that, in a symmetric equilibrium, each limit-order trader receives the quantity $q = -y/N$. Given the equilibrium demand schedule f, the equilibrium clearing price, p, is the root of $f(p) = -y/N$. We conclude the equilibrium price is simply the inverse of the equilibrium demand schedule function evaluated at $q = -y/N$. Note that the inverse of f is the solution of the linear o.d.e.:

$$qp'(q) = (N-1)[p - b(-Nq)] \tag{22}$$

with the boundary condition $p(\infty) = -U$, $p(-\infty) = U$. We denote the solution to (22) by $p_o(q)$ and conclude that the equilibrium clearing price in the open-book environment is $p_o(-\tilde{y}/N)$.[18]

We now turn our attention to the closed-book environment. The expected profit of a limit-order trader is

$$E[(\tilde{v} - \tilde{p}) f_i(\tilde{p}) | m_i = 1], \quad \text{where } f_i(\tilde{p}) + f_{-i}(\tilde{N}, \tilde{p}) + \tilde{y} = 0 \tag{23}$$

and m_i is the information available to the ith trader, namely, that the book contains his order. An equilibrium consists of demand schedules f_i for each of the limit-order traders, such that, for each i, f_i maximizes (23) over the class of continuously differentiable functions. A symmetric equilibrium is an equilibrium in which all limit-order traders submit the same demand schedule f.

THEOREM 8. Let f form a symmetric equilibrium. Then, f satisfies the equation

$$\frac{f(p)}{f'(p)} = E\{(\tilde{N} - 1)[\tilde{p} - b(\tilde{y})] | \tilde{p} = p\}; \; f(U) = -\infty, \; f(-U) = \infty, \tag{24}$$

where U is the upper bound of the support of $\tilde{v}$, and the joint distribution of $\tilde{p}$, $\tilde{y}$, and $\tilde{N}$ is defined via the clearing equation:

$$\tilde{N} f(p) = -y. \tag{25}$$

The proof of the theorem is given in Appendix E. Note that equation (21), which describes the equilibrium demand schedule in the open-book environment, is a special case of (24) when N is known. To

18. In the standard Kyle (1985) model, the equilibrium clearing price is λy for some constant λ. Here, the clearing price is a function of y and the number of limit-order traders, N.

transform (24) into a proper o.d.e., we use the equilibrium clearing condition (25) to express the conditional expectation. Again, we find it more convenient to express the o.d.e. in terms of the inverse demand schedule. We write $f(p) = q$. The quantity $f(\tilde{p})$ is informationally equivalent to $\tilde{p}$. Furthermore, in a symmetric equilibrium, $f(\tilde{p}) = -\tilde{y}/\tilde{N}$. Hence $f(p)$ is the equilibrium demand schedule only if its inverse $p(q)$ solves

$$qp'(q) = E[(\tilde{N} - 1)(p(q) - b(\tilde{y}))| - \tilde{y}/\tilde{N} = q] \tag{26}$$

with the boundary condition $p(\infty) = -U$, and $p(-\infty) = U$

Let

$$\begin{aligned} h_1(q) &= E[\tilde{N} - 1| - \tilde{y}/\tilde{N} = q] \\ h_0(q) &= E[b(\tilde{y})(\tilde{N} - 1)| - \tilde{y}/\tilde{N} = q] \end{aligned}$$

then (26) can be written as an o.d.e.:

$$p'(q) = ph_1(q)/q - h_0(q)/q. \tag{27}$$

Lemma 3 in the Appendix E implies

$$h_1(q) = \frac{\sum_{n=1}^{K}(n-1)ng(-nq)P_n}{\sum_{n=1}^{K} ng(-nq)P_n}$$

and

$$h_0(q) = E[b(-\tilde{N}q)(\tilde{N}-1)|\tilde{y}/\tilde{N} = q] = \frac{\sum_{n=1}^{K} b(-nq)(n-1)ng(-nq)P_n}{\sum_{n=1}^{K} ng(-nq)P_n},$$

where g is the density of y, K is the upper bound on the support of $\tilde{N}$, and $P_n = \text{Prob}(N = n)$. We donote the solution to the o.d.e. (27) by $p_c(q)$. The function $p_c(q)$ is the inverse function of the equilibrium demand schedule. Hence, the equilibrium clearing price in a closed-book environment is $p_c(-\tilde{y}/\tilde{N})$.

Next, we compare equilibria. In both environments, the equilibrium clearing price is a zero-mean random variable that takes positive values if and only if the market-order imbalance y is positive. Also, by assumption, the informativeness of market orders is identical in both markets. Thus, our analysis focuses on the rents liquidity providers extract.

Because prices in the unrestricted model are nonlinear, we have no simple measure of liquidity. We have, in our model, that $E[p|y = 0] = 0$ in both environments. Thus, for a given market order y, the price impact of y is $E[p|y]/y$. The next theorem shows that, for small market orders, the open-book environment provides better liquidity (price impacts are smaller). Recall that, given a market order y, the clearing prices are $p_0(y/N)$ and $p_c(y/N)$ in the open and closed environments, respectively.

THEOREM 9. Given a sufficiently small market-order imbalance, the clearing price in the open-book environment on average, is closer to zero than in the closed-book environment. That is, there is an $\varepsilon > 0$ such that, for all $|y| \in (0, \varepsilon)$,

$$0 < \frac{E[p_o(\tilde{y}/\tilde{N})|\tilde{y} = y]}{y} < \frac{E[p_c(\tilde{y}/\tilde{N})|\tilde{y} = y]}{y}.$$

The market-order imbalance y and the number of limit-order traders N are independent. Thus, given y, the average clearing price in the open- and closed-book environments are $Ep_o(-y/\tilde{N})$ and $Ep_c(-y/\tilde{N})$, respectively, where the expectation is taken over the random variable $\tilde{N}$ (see example 1.5 in Durret 1996, p. 224). For every function $p(\cdot)$ that vanishes at zero, we can write

$$Ep(-y/\tilde{N}) = \int_0^y E\frac{-1}{\tilde{N}}p'(-s/\tilde{N})ds$$

We define the functions $\varphi_c(s) = E\frac{-1}{\tilde{N}}p'_c(-s/\tilde{N})$ and $\varphi_o(s) = E\frac{-1}{\tilde{N}}p'_o(-s/\tilde{N})$. Using l'Hospital rule, we show (details are in Appendix E)

$$\varphi_o(0) = E\frac{1}{N}\frac{(N-1)N}{N-2}b'(0)$$
$$\varphi_c(0) = E\frac{1}{N}\frac{EN^2(N-1)}{EN(N-2)}b'(0).$$

We then show that, regardless of the distribution of $\tilde{N}$, $\varphi_o(0) < \varphi_c(0)$. This implies that there is an $\varepsilon > 0$ such that $\varphi_o(s) < \varphi_c(s)$ for all $s \in (-\varepsilon, \varepsilon)$. Hence, the statement in the theorem follows.

We cannot show that, in the open-book environment, price impacts are smaller for all sizes of market orders. In fact, in the examples we explicitly solve, we find that for large orders the price impacts are smaller in the closed-book environment. We therefore study two alternative measures of average market liquidity. The first measure is simply $E\,|\,p|$, and the second measure is the expected gain of a liquidity provider. Next, we consider two examples, and we find that both $E\,|p|$ and the expected gains of a limit-order trader (i.e., a liquidity provider) are smaller in the open-book environment. We also find that the variance of the clearing price is smaller in the open-book environment.

In both examples, the number of traders is either low (10) with probability 1/2 or high (15) with probability 1/2. Also, $y = v + z$, where z is a standard normal random variable. In the first example, the liquidation

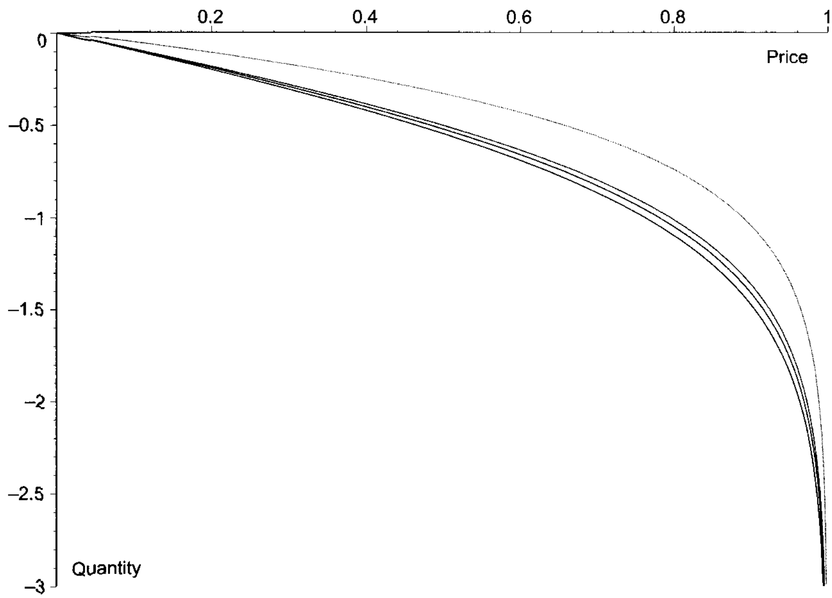

FIG. 5.—The equilibrium sell side of the limit-order book in an open-book environment. In this example, the liquidation value takes the values -1 or 1 with probability 1/2. The curves are the sell side of the limit-order book in four cases. The upper curve corresponds to $N = 3$ and, in descending order, $N = 13$, $N = 23$, and $N = \infty$.

value, v, takes the values -1 or 1 with probability $1/2$. Thus, the density $g(y)$ of y is

$$g(y) = \frac{1}{2}\phi(y+1) + \frac{1}{2}\phi(y-1),$$

where $\phi(\cdot)$ is the density of a standard normal random variable, and the conditional expectation is

$$b(y) = E[v|y] = \frac{\phi(y-1) - \phi(y+1)}{\phi(y-1) - \phi(y+1)}.$$

Figure 5 shows the equilibrium demand schedule when the book is closed as well as the equilibrium demand schedule when the book is open. In the open-book environment, limit-order traders adjust their demand schedules to what they see in the book. The more limit-order traders participate, the less liquidity each one of them provides. Figure 6 shows the realization of the book. The book is random in both environments. Because limit-order traders compete effectively in the open-book environment, the realized book is not sensitive to the realized number of traders. In fact, limit-order traders compete so effectively that, if we

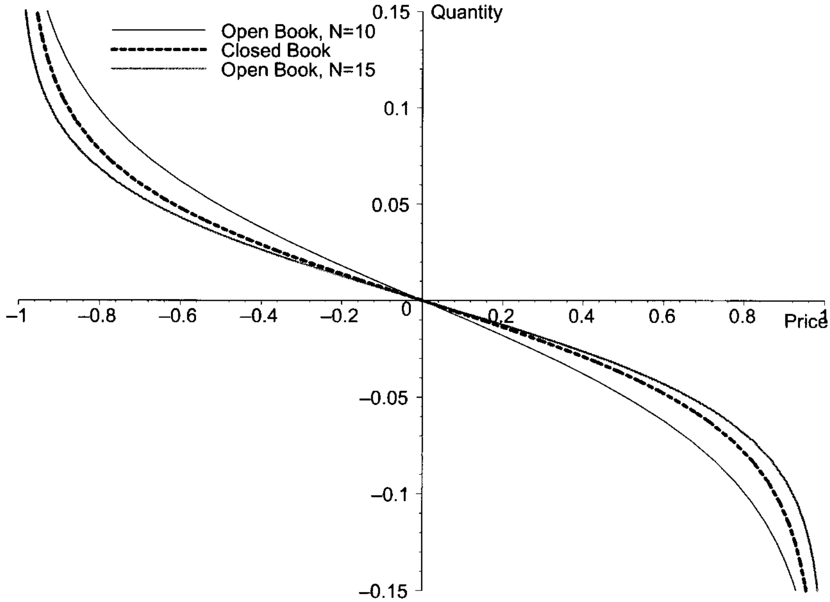

FIG. 6.—Equilibrium demand schedules when the liquidation value takes one of two values.

slightly increase the number of traders, the realized book in the open-book environment is hardly distinguishable from the limiting case of infinite number of traders (i.e., the competitive case). In contrast, in a closed-book environment, the realized limit-order book is very sensitive to the realized number of traders. Thus, we are not surprised to find that, in an open-book environment, prices are less volatile. Indeed, in our example (with a relatively small number of traders), the variance of clearing prices are 0.550 (competitive case), 0,582 (open book), and 0.590 (closed book). The expected absolute clearing prices are 0.683 (competitive case), 0.706 (open-book environment), and 0.714 (closed-book environment). The expected gain of a liquidity provider (a limit-order trader) is 0 (competitive case), 0.0018 (open book), and 0.0024 (closed book).

We now consider a second example, in which v is a standard normal random variable, so that $b(y) = \frac{1}{2}y$. Figure 7 shows the optimal demand schedule. Note that, in the open-book environment (as in Kyle 1989), the equilibrium book is linear. Figure 8 shows the realized book, which is again linear in the open-book environment. We find that the variance of clearing prices are 0.500 (competitive case), 0.604 (open book), and 0.612 (closed book). The expected absolute clearing prices are $\sqrt{1/\pi} \sim$ 0.564 (competitive case), 0.619 (open-book environment), and 0.635 (closed-book environment). The expected gain of a liquidity provider

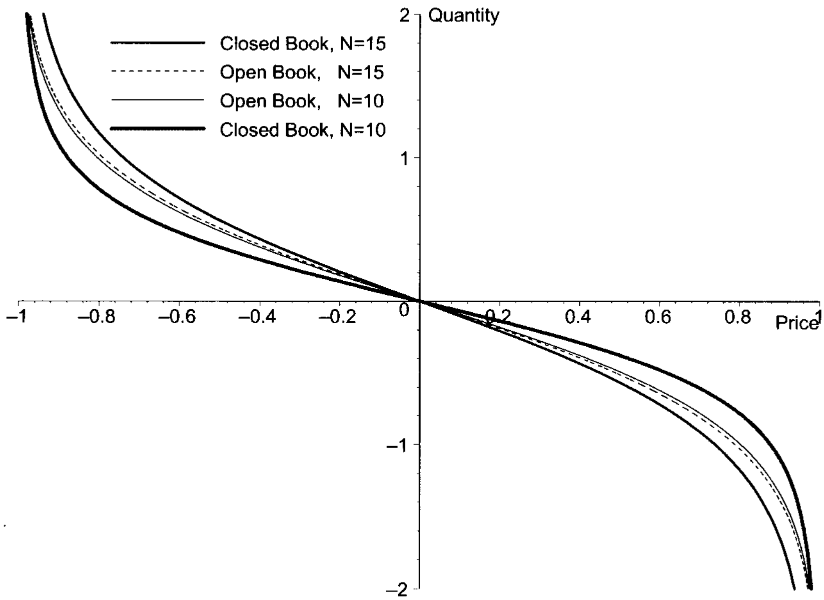

FIG. 7.—Equilibrium limit-order book when the liquidation value takes one of two values.

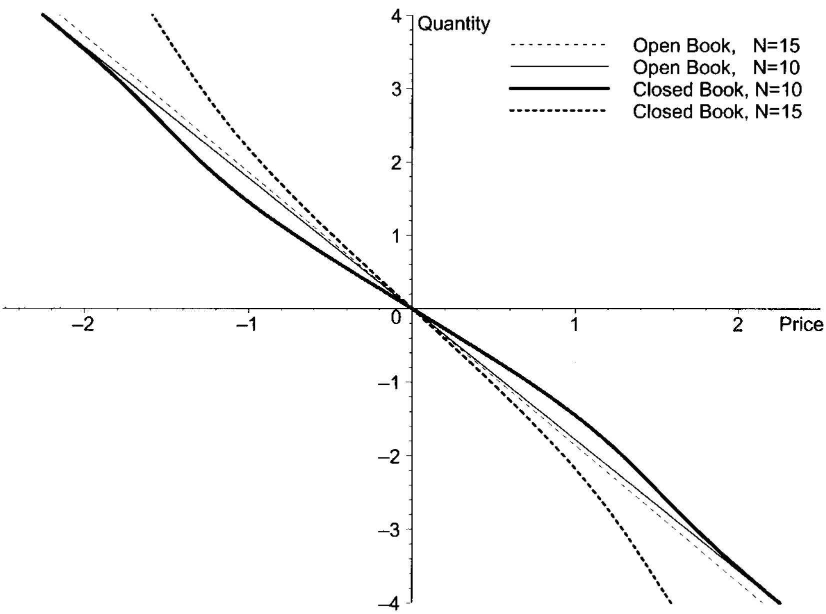

FIG. 8.—Equilibrium demand schedules when the liquidation value is normally distributed.

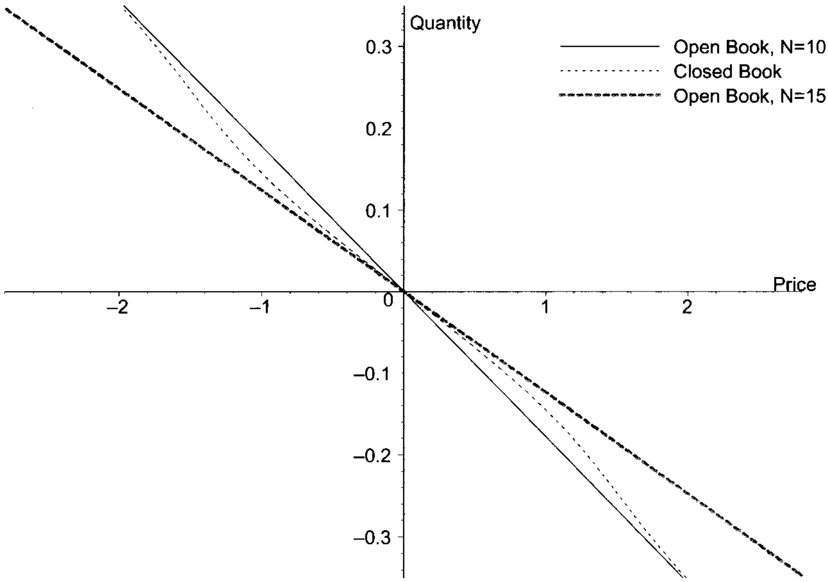

FIG. 9.—Equilibrium limit-order when the liquidation value is normally distributed

(a limit-order trader) is 0 (competitive case), 0.0008 (open book), and 0.001 (closed book).

In both examples, the market is less volatile and on average more liquid when the book is open. However, in both examples, when we consider large market orders, the price impact of market order is smaller in the closed-book environment. Indeed, we can infer from graphs of the realized book (see figures 6 and 8) the clearing price as a function of the market order imbalance. The vertical axis (marked Quantity) is the amount the limit-order traders absorb at a given price. This amount has the opposite sing of the market-order imbalance. Given a market-order imbalance, y, the realized clearing price is the price at which an horizontal line at $-y$ intercepts the realized book. The average clearing price in an environment is its simple average of the clearing prices (because in these examples, the number of limit-order traders has the same probability of being high or low). The figures demonstrate the result in theorem 9: for small orders, the average clearing price is closer to zero when the book is open. However, interestingly, we also see that the opposite is true for large orders.

For large orders, the limit-order traders provide too much liquidity in the closed-book environment. In fact, conditioned on a large order imbalance, limit-order traders lose money in the closed-book environment. This happens in equilibrium because limit order traders can condition only on prices, not on aggregate order imbalances. Conditioned on prices, their expected profit is always positive. A high clearing price implies that either the number of limit-order traders is small (in that case, limit-order traders extract high rents) or the market order imbalance is high (in that case, limit-order traders trade against informed traders). Competition drives down limit-order traders' expected gain, and on average, this can happen only if they lose when market-order imbalance is high.

VII. Concluding Remarks

This paper compares a specialist call market in which the limit-order book is closed to one in which each trader observes the book. Our model captures the informational advantage the liquidity providers (specialist and the limit-order traders) have in a closed book environment. Our results demonstrate that removing these informational advantages by opening the book reduces the liquidity providers' market power. More specific, we show that, on average, the traders who demand immediacy benefit from opening the book, while the traders who supply immediacy prefer a closed-book environment. We also show that, on average, prices in an open-book environment are more informative.

Appendix A

Proof of Theorem 1

This section is devoted to the closed book environment. To save on notation, we omit writing the subscript c.

The proof is divided into three parts. In the first part, we analyze the specialist clearing price, taking the traders' strategies as given. In the second part, we derive the the limit-order traders' optimal demand schedule, given the clearing price rule and the informed trader's strategy. In the third part, given the clearing price rule, we derive the informed trader's optimal market order.

The Specialist

The specialist observes a linear book $f(p) = a + (\bar{v} - p)A$ and aggregate market order. He chooses a clearing price to maximize his expected gain. The normality assumption together with the linear form of the informed trader's market order implies that

$$E[\tilde{v}|\tilde{y}] = \bar{v} + b\tilde{y},$$

where b is defined via the first equation in (7). The specialist's problem is

$$\max_p (p - \bar{v} - by)[y + a + A(\bar{v} - p)].$$

The solution is

$$P(y, f) = \bar{v} + \frac{1}{2}\left(b + \frac{1}{A}\right)\tilde{y} + \frac{1}{2}\frac{a}{A}, \tag{A1}$$

where $A = -f'$ and $a = f(\bar{v})$. The quantity that the specialist absorbs under the optimal clearing price rule is

$$\frac{1}{2}(bA-1)\tilde{y} - \frac{1}{2}a. \tag{A2}$$

In the next section, we prove that, given this price rule, the book has the form $(v-p)\tilde{N}B$; that is, $a = 0$. This implies that we can also express the clearing price as

$$\tilde{p} = \bar{v} + \lambda\tilde{y},$$

where $\lambda = \frac{1}{2}(b + \frac{1}{\tilde{N}B})$, and this proves the necessity of the first and second equations in (7).

The Limit-Order Traders

Let $f_{-1} = (\bar{v} - p)C(\tilde{N})$, and let the first trader's demand schedule be $(\bar{v} - p)B + a$, where B and a are arbitrary constants. First, we show that it is not optimal to submit $a \neq 0$. From (A1), it follows that the clearing price is

$$\tilde{p} = \bar{v} + \frac{1}{2}\left(b + \frac{1}{C(\tilde{N}) + B}\right)\tilde{y} + \frac{1}{2}\frac{a}{C(\tilde{N}) + B}.$$

The constant a can be viewed as a market order, however, one that contains no information. From the specialist's optimal clearing price it follows that, on average, the specialist is on the other side of this order; therefore, it cannot be optimal. Formally, let $\tilde{p}_0$ and V_0 be the clearing price and trader's expected gain, respectively, if the trader submits $(\bar{v} - p)B$; that is, $a = 0$. Since the informed trader's linear strategy implies that $E[\tilde{v} - p_0|m] = 0$, the expected profit from the demand schedule $(\bar{v} - p)B + a$ is

$$\begin{aligned}
&E\left\{\left(\tilde{v} - \tilde{p}_0 - \frac{1}{2}\frac{a}{C(\tilde{N}) + B}\right)\left[(\bar{v} - \tilde{p}_0)B - \frac{1}{2}\frac{a}{C(\tilde{N}) + B}B + a\right]\middle| m_i = 1\right\} \\
&= V_0 - E\left[\frac{1}{2}\frac{a^2}{B + C(\tilde{N})}\left(1 - \frac{B}{2(C(\tilde{N}) + B)}\right)\middle| m_i = 1\right] \\
&\leq V_0.
\end{aligned}$$

We conclude that a has to be zero.

In the following, we take $a = 0$ and solve for the optimal slope B. While the trader cannot observe the market-order imbalance directly, he can infer some information about if from the clearing price. The inverse relation between the clearing price and the market-order imbalance is given by

$$\tilde{y} = 2\frac{C(\tilde{N}) + B}{1 + b[C(\tilde{N}) + B]}(\tilde{p} - \bar{v}),$$

and we conclude that

$$E[\tilde{v}|\tilde{y}, m_i = 1] = \bar{v} + b\tilde{y} = \bar{v} + 2b\frac{C(\tilde{N}) + B}{1 + b[C(\tilde{N}) + B]}(\tilde{p} - \bar{v}).$$

where in the second equality b is defined via the first equation in (7). The trader's objective function is

$$\begin{aligned}
&E[(\tilde{v} - \tilde{p})x \,|\, m_i = 1] = E[(\tilde{v} - \tilde{p})(\tilde{v} - \tilde{p})B \,|\, m_i = 1] \\
&= E[E[(\tilde{v} - \tilde{p})(\bar{v} - \tilde{p})B \,|\, \tilde{y}, m_i = 1] \,|\, m_i = 1] \\
&= E\left\{\left[\bar{v} + 2b\frac{C(\tilde{N}) + B}{1 + b(C(\tilde{N}) + B)}(\tilde{p} - \bar{v}) - \tilde{p}\right](\bar{v} - \tilde{p})B \,\middle|\, m_i = 1\right\} \\
&= E\left(\left\{1 - 2b\frac{C(\tilde{N}) + B}{1 + b[C(\tilde{N}) + B]}\right\}(\bar{v} - \tilde{p})^2 B \,\middle|\, m_i = 1\right) \\
&= E\left(\left\{1 - 2b\frac{C(\tilde{N}) + B}{1 + b[C(\tilde{N}) + B]}\right\}\left\{\frac{1}{2}\left[b + \frac{1}{C(\tilde{N}) + B}\right]\tilde{y}\right\}^2 B \,\middle|\, m_i = 1\right) \\
&= E\left(\left\{1 - 2b\frac{C(\tilde{N}) + B}{1 + b[C(\tilde{N}) + B]}\right\}\left\{\frac{1}{2}\left[b + \frac{1}{C(\tilde{N}) + B}\right]\right\}^2 B \,\middle|\, m_i = 1\right) E[\tilde{y}^2 | \, m_i = 1] \\
&= E\left(\left\{\frac{1 - b[C(\tilde{N}) + B]}{1 + b[C(\tilde{N}) + B]}\right\}\left\{\frac{1}{2}\frac{1 + b[C(\tilde{N}) + B]}{C(\tilde{N}) + B}\right\}^2 B \,\middle|\, m_i = 1\right) E\tilde{y}^2 \\
&= \frac{1}{4}E\left(\frac{\{1 - b[C(\tilde{N}) + B]\}\{1 + b[C(\tilde{N}) + B]\}}{[C(\tilde{N}) + B]^2} B \,\middle|\, m_i = 1\right) E\tilde{y}^2 \\
&= \frac{1}{4}E\left(\left\{\frac{B}{[C(\tilde{N}) + B]^2} - b^2 B\right\} \,\middle|\, m_i = 1\right) E\tilde{y}^2.
\end{aligned}$$

Since $\tilde{N}$ is finite, we can take the derivative under the expectation operator. The first- and second-order conditions are given by

$$E\left\{\frac{C(\tilde{N}) - B}{[C(\tilde{N}) + B]^3} \,\middle|\, m_i = 1\right\} = b^2$$

$$2E\left\{\frac{-2C(\tilde{N}) + B}{[C(\tilde{N}) + B]^4} \,\middle|\, m_i = 1\right\} < 0,$$

respectively. In a symmetric equilibrium $C(\tilde{N}) = (\tilde{N} - 1)B$. Hence, the second-order condition holds and the optimal slope B is the root of

$$\frac{1}{B^2} E\left[\frac{(\tilde{N} - 2)}{\tilde{N}^3} \middle| m_i = 1 \right] = b^2.$$

The positive root is given by

$$B = \frac{1}{b} \sqrt{E\left[\frac{\tilde{N} - 2}{\tilde{N}^3} \middle| m_i = 1 \right]}.$$

This proves the necessity of the third equation in (7).

The Informed Trader

The informed trader affects the prices through the market-order imbalance. Assume that the clearing price is given by $p = \bar{v} + \lambda(\tilde{N})y$ for some positive function $(\lambda \tilde{N})$. The trader's problem is

$$\begin{array}{ll} \max\limits_{x} & E(v - \tilde{p})x \\ \text{such that} & p = \bar{v} + \tilde{\lambda}\tilde{y}. \end{array}$$

Relying on the independence of $\tilde{z}$, $\tilde{v}$, and $\tilde{N}$, we can rewrite the problem as

$$\max_{x} (v - \bar{v})x - x^2 E\tilde{\lambda}(\tilde{N}).$$

The optimal solution is given by

$$x = \frac{1}{2\tilde{\lambda}}(v - \bar{v}),$$

and it implies the fourth equation in (7).

Appendix B

Limit on Number of Traders

To see that $N \geq 8$ is not necessary to obtain the results in lemma 2, we consider the following family of truncated binomial distributions parametrized by p:

$$\text{Prob}(N = i) = \frac{\binom{7}{i} p^i (1 - p)^{7-i}}{\sum_{j=3}^{7} \binom{7}{j} p^j (1 - p)^{7-j}}, \quad i = 3, \ldots, 7.$$

We compared $E\tilde{\lambda}_c$ with $E\tilde{\lambda}_o$ for different p, under the assumption that $2\sigma_v = \sigma_z$. The results, shown in figure A1, demonstrate that even in the case that the whole support of N lies below 8, the results of lemma 2 can still hold.[19]

Appendix C

Proof of Theorem 5

To calculate the limit-order traders' expected profit, we use the independence of $\tilde{z}$, $\tilde{v}$, and $\tilde{N}$ (hence, $\tilde{\lambda}$). In the closed-book environment, B_c, b_c, and β_c are constants. We use the second equation in (7) to conclude that aggregated expected profit in a closed-book environment is given by

$$\begin{aligned}
&E\tilde{N}(\tilde{v}-\tilde{p})(\bar{v}-\tilde{p})B_c\\
&= E\tilde{N}\{\tilde{v}-\bar{v}-\tilde{\lambda}_c[\tilde{z}+\beta_c(\tilde{v}-\bar{v})]\}\{-\tilde{\lambda}_c[\tilde{z}+\beta_c(\tilde{v}-\bar{v})]\}B_c\\
&= E\tilde{N}\left[-\sigma_v^2\tilde{\lambda}_c\beta_c+\tilde{\lambda}_c^2\left(\sigma_z^2+\beta_c^2\sigma_v^2\right)\right]B_c\\
&= -B_c\sigma_v^2\beta_c E\left[\frac{\tilde{N}}{2}\left(b_c+\frac{1}{\tilde{N}B_c}\right)\right]+B_c\left(\sigma_z^2+\beta_c^2\sigma_v^2\right)E\left[\frac{\tilde{N}}{4}\left(b_c^2+\frac{1}{\tilde{N}B_c}\right)^2\right]\\
&= -\frac{B_c}{2}\sigma_v^2\beta_c\left(b_cE\tilde{N}+\frac{1}{B_c}\right)+\frac{B_c}{4}\left(\sigma_z^2+\beta_c^2\sigma_v^2\right)\left(b_c^2E\tilde{N}+2\frac{b_c}{B_c}+\frac{1}{B_c^2}E\frac{1}{\tilde{N}}\right)\\
&=: g\left(E\tilde{N},E\tilde{a},E\tilde{r}\right),
\end{aligned}$$

where the definition of g is possible, since we can express B_c, b_c, and β_c in terms of $E\tilde{N}$, $E\tilde{a}$, and $E\tilde{r}$ according to (15).

We calculate the aggregate expected profit in the open-book equilibrium in a similar way. We use second equation in (13) to express $\tilde{\lambda}_o$. However, we note that B_o, b_o, and β_o are all random and hence cannot be taken outside the expectation operator:

$$\begin{aligned}
&E\tilde{N}(\tilde{v}-\tilde{p})(\bar{v}-\tilde{p})B_o\\
&= E\tilde{N}\{\tilde{v}-\bar{v}-\tilde{\lambda}_o[\tilde{z}+\beta_o(\tilde{v}-\bar{v})]\}\{-\tilde{\lambda}_o[\tilde{z}+\beta_o(\tilde{v}-\bar{v})]\}B_o\\
&= E\tilde{N}\left[-\sigma_v^2\tilde{\lambda}_o\beta_o+\tilde{\lambda}_o^2\left(\sigma_z^2+\beta_o^2\sigma_v^2\right)\right]B_o\\
&= E\left\{-B_o\sigma_v^2\beta_o\left[\frac{\tilde{N}}{2}\left(b_o+\frac{1}{\tilde{N}B_o}\right)\right]+B_o\left(\sigma_z^2+\beta_o^2\sigma_v^2\right)\left[\frac{\tilde{N}}{4}\left(b_o^2+\frac{1}{\tilde{N}B_o}\right)^2\right]\right\}\\
&= E\left[-\frac{B_o}{2}\sigma_v^2\beta_o\left(b_o\tilde{N}+\frac{1}{B_o}\right)+\frac{B_o}{4}\left(\sigma_z^2+\beta_o^2\sigma_v^2\right)\left(b_o^2\tilde{N}+2\frac{b_o}{B_o}+\frac{1}{B_o^2}\frac{1}{\tilde{N}}\right)\right]\\
&= g\left(\tilde{N},\tilde{a},\tilde{r}\right)
\end{aligned}$$

It takes some algebraic simplifications to show that the function g can be expressed as in (16). To prove that $Eg(\tilde{N},\tilde{a},\tilde{r}) \leq g(E\tilde{N},E\tilde{a},E\tilde{r})$, we can assume without loss of generality that $\sigma_v\sigma_z/4 = 1$. It is convenient to express the function g as

$$g \equiv \sqrt{h_1} \times \sqrt{\sqrt{h_2} \times \sqrt{h_3}}$$

19. Note that, in the cases $p = 0$ and $p = 1$, the distribution that governs the noise is degenerate and, as a result, the two equilibria are identical.

where

$$h_1(a, r) = \frac{(a - r)}{a\sqrt{r}}$$
$$h_2(N, a, r) = (a - r)N$$
$$h_3(a, r) = (a - r).$$

We need to show that

$$Eg\left(\tilde{N}, \tilde{a}, \tilde{r}\right) \leq \sqrt{h_1(E\tilde{a}, E\tilde{r})}\sqrt{\sqrt{h_2\left(E\tilde{N}, E\tilde{a}, E\tilde{r}\right)}\sqrt{h_3(E\tilde{a}, E\tilde{r})}}.$$

Using a Cauchy-Schwartz inequality, we have:

$$\begin{aligned} Eg\left(\tilde{N}, \tilde{a}, \tilde{r}\right) &\leq \sqrt{Eh_1(\tilde{a}, \tilde{r})}\sqrt{E\left\{\sqrt{h_2\left(\tilde{N}, \tilde{a}, \tilde{r}\right)}\sqrt{h_3(\tilde{a}, \tilde{r})}\right\}} \\ &\leq \sqrt{Eh_1(\tilde{a}, \tilde{r})}\sqrt{\sqrt{Eh_2\left(\tilde{N}, \tilde{a}, \tilde{r}\right)}\sqrt{Eh_3(\tilde{a}, \tilde{r})}} \end{aligned}$$

To see that $Eh_1(\tilde{a}, \tilde{r}) \leq h_1(E\tilde{a}, E\tilde{r})$, we note that

1. The function $a \rightarrow h_1[a, r(a)]$ is concave on the interval (0,1/8).
2. The function h_1 (a, r) is decreasing with r.
3. The function $r(a)$ is concave.

Hence,

$$Eh_1(\tilde{a},\tilde{r}) \leq h_1[E\tilde{a}, r(E\tilde{a})] \leq h_1(E\tilde{a}, E\tilde{r}).$$

To see that $Eh_2(\tilde{N}, \tilde{a}, \tilde{r}) \leq h_2(E\tilde{N}, E\tilde{a}, E\tilde{r})$, we use the following

1. The function $a \rightarrow h_2[N(a), a, r(a)]$ is linear.
2. The function h_2 $[N, E\tilde{a}, r(E\tilde{a})]$ is increasing in the first argument.
3. The function $N(a)$ is convex.
4. The function h_2 (a, r) is decreasing with r.
5. The function $r(a)$ is concave.

It follows that

$$Eh_2\left(\tilde{N}, \tilde{a}, \tilde{r}\right) \leq h_2[N(E\tilde{a}), E\tilde{a}, r(E\tilde{r})] \leq h_2\left[E\tilde{N}, E\tilde{a}, r(\tilde{a})\right] \leq h_2\left(E\tilde{N}, E\tilde{a}, E\tilde{r}\right).$$

The function h_3 is linear, and hence, $Eh_3(\tilde{a}, \tilde{r}) = h_3(E\tilde{a}, E\tilde{r})$. This ends the proof.

Appendix D

Proof of Lemma 6

In equilibrium, the market-order imbalance has the form $\beta(\tilde{v} - \bar{v}) + \tilde{z}$, where β is a constant (which, in the open-book environment, depends on the commonly known number of limit-order traders). Since $\tilde{v}$ and $\tilde{z}$ are independent and normal, it follows that

$$\operatorname{var}(\tilde{v} \mid \tilde{y}) = \frac{\sigma_z^2 \sigma_v^2}{\beta^2 \sigma_v^2 + \sigma_z^2}.$$

We have

$$E \frac{\sigma_z^2 \sigma_v^2}{\beta_o^2 \sigma_v^2 + \sigma_z^2} = E \frac{4\lambda_0^2 \sigma_z^2 \sigma_v^2}{\sigma_v^2 + 4\lambda_o^2 \sigma_z^2}.$$

The function $x\!-\!> 4x^2\sigma_z^2\sigma_v^2/(\sigma_v^2 + 4x^2\sigma_z^2)$ is concave on the interval $[1/\sqrt{12} \times \sigma_v/\sigma_z,\ \infty)$. It is also increasing. From the solution of λ given in (15), we know that λ_0 is always greater than $1/\sqrt{12}\sigma_v/\sigma_z$. Thus,

$$E \frac{4\lambda_0^2 \sigma_z^2 \sigma_v^2}{\sigma_v^2 + 4\lambda_o^2 \sigma_z^2} \le \frac{4E(\lambda_o)^2 \sigma_z^2 \sigma_v^2}{\sigma_v^2 + 4(E\lambda_o)^2 \sigma_z^2} \le \frac{4E(\lambda_c)^2 \sigma_z^2 \sigma_v^2}{\sigma_v^2 + 4E(\lambda_c)^2 \sigma_z^2} = \frac{\sigma_z^2 \sigma_v^2}{\beta_c^2 \sigma_v^2 + \sigma_z^2},$$

where, for the second inequality, we used the relation $E\lambda_0 \le E\lambda_c$ from lemma 2. This proves that[20]

$$E \operatorname{var}(\tilde{v} | \tilde{y}_o) \operatorname{var}(\tilde{v} \mid \tilde{y}_c).$$

In the open-book environment, there is a one-to-one relation between the opening price and the order imbalance. We therefore have $E \operatorname{var}(\tilde{v} \mid \tilde{y}_o) = E \operatorname{var}(\tilde{v} \mid \tilde{p}_o)$. Thus, to end the proof, it is enough to show that $\operatorname{var}(\tilde{v} \mid \tilde{y}_c) \le E \operatorname{var}(\tilde{v} \mid \tilde{p}_c)$. Note that $\operatorname{var}(\tilde{v} \mid \tilde{y}_c) = \operatorname{var}(\tilde{v} \mid \tilde{y}_c, \tilde{p}_c)$; that is, given $\tilde{y}$, the clearing price does not add new information. Thus, our result follows from a fact that the conditional expectation minimizes the mean square root.[21]

Appendix E

Proof of Theorems 7–9

Proof of Theorem 7

The number of traders N is known, so we view it as a parameter. The only source of uncertainty is the market-order imbalance y. We focus on the problem of the first

20. In the closed-book environment, $\operatorname{var}(\tilde{v} \mid \tilde{y}_c)$ is a constant.
21. If var $(x) < \infty$, then

$$E \operatorname{var}(x|y) = E\Big(x - E(x|y)^2\Big) = \min_{z \varepsilon \sigma(y)} E(x - z)^2,$$

where $\sigma(y)$ is the sigma-algebra generated by y (see Durret 1996, p. 227).

trader, taking the demand schedule of the other traders as given. We are looking for a symmetric equilibrium in which the limit-order traders submit is monotone. We therefore can replace the problem of the first limit-order trader with an artificial problem in which he submits a y-contingent order, $h(y)$, that maximizes[22]

$$E(v-p)h(y), \quad \text{where} \quad h(y)+(N-1)f(p)+y=0.$$

Because we are looking for an equilibrium in which $f(p)$ is monotone, $f(p)$ should have an inverse $p(q)$. We therefore can rewrite the objective of the artificial problem without the side condition:

$$E\left\{v-p\left[-\frac{y-h(y)}{N-1}\right]\right\}h(y)=E\left\{b(y)-p\left[\frac{-y-h(y)}{N-1}\right]\right\}h(y), \qquad \text{(A3)}$$

where for the second equality, we use the law ofiterated conditional expectation. The advantage of using the artificial problem is that we know its solution in equilibrium. The solution has to be the function $-y/N$. Thus, to find an equilibrium, we are looking for a function $p(\cdot)$, such that the maximum of (A3) is attained at the function $h_0(y)\equiv -y/N$.

For every y, define the function

$$\phi(y,q)=\left\{b(y)-p\left[-\frac{(y+q)}{N-1}\right]\right\}q.$$

Clearly, if for all y and for every function $h(y)$, we have $\phi[y,h(y)]\leq\phi[y,h_0(y)]$, then $h_0(y)$ is optimal.

This is the case when each of the functions $\phi(y,\cdot)$ attains its maximum at $q=-y/N$. Thus, we want the first-order condition to hold at $q=-y/N$:

$$0=\phi_q(y,-y/N)=\left[b(y)-p\left(\frac{-y}{N}\right)\right]+\frac{1}{(N-1)}\frac{y}{N}p'\left(-\frac{y}{N}\right).$$

For the first-order condition to hold, the function $p(\cdot)$ has to satisfy the linear o.d.e.:

$$0=[b(-Nq)-p(q)]-\frac{1}{(N-1)}qp'(q). \qquad \text{(A4)}$$

We add the natural boundary conditions $p(\infty)=U$ and $p(-\infty)=-U$, where U is the upper bound of the support of $\tilde{v}$.

From now on, we assume p solves the o.d.e. (4). We next have to show that the first-order condition is sufficient. It is straightforward to verify that the second-order condition holds; that is, $\phi_{qq}(y,-y/N)<0$. To show that $q=-y/N$ is a global

22. Indeed, let the demand schedule of the other traders be given. For any price-contingent order $g(p)$, the clearing condition defines the price as a function of the market-order imbalance, $p_g(y)$. Thus, the y-contingent order $h(y)=g[p_g(y)]$ attains the same gains as the price-contingent order $g(p)$. We show that the gains of the optimal y-contingent order can be attained using a price-contingent order.

maximum of $\phi(y, \cdot)$, we use the following argument. We denote the global maximum by $q(y)$. Because $b(0) = 0$ and the solution p to the o.d.e. (A4) is strictly decreasing and satisfies $p(0) = 0$, we can show $\phi(0, q) < 0$ for all $q \neq 0$. Hence, $q(0) = 0$. Differentiating the first-order condition, we get an o.d.e. that the global solution, $q(y)$, must satisfy:

$$q'(y) = -\frac{\phi_{qy}(y, q)}{\phi_{qq}(y, q)}, \qquad q(0) = 0 \tag{A5}$$

Because, by construction, $-y/N$ solves the first-order condition, it also solves the o.d.e. (A5). Moreover, under the condition in the theorem, the o.d.e. (5) has a unique solution. Hence $q = -y/N$ is a global maximum of $\phi(y, \cdot)$.

We have concluded that if $p(\cdot)$ solves the o.d.e. (A4) then the y-contingent order $h_0(y) = -y/N$ is optimal. Now, let f satisfy the condition in the theorem (i.e., f satisfies [21]). Then its inverse satisfies (A4). Hence, f defines a symmetric equilibrium.

Proof of Theorem 8
Consider the problem of the ith limit-order trader. Let $f_{-i} = (\tilde{N} - 1)f$ be given. The demand schedule $h(p)$ is optimal only if, for every demand schedule $k(p)$, we have $J'(0) = 0$, where $J(\varepsilon)$ is given by

$$J(\varepsilon) = E_i(\tilde{v} - p)[h(p) + \varepsilon k(p)] \text{ and } h(p) + \varepsilon k(p) + \left(\tilde{N} - 1\right)f(p) + y = 0.$$

Thus, p is an implicit function of ε that satisfies

$$p_\varepsilon = \frac{-k}{h' + \varepsilon k' + \left(\tilde{N} - 1\right)f'}.$$

Hence, if $h(p)$ is optimal, we must have

$$0 = J'(0) = E_i(\tilde{v} - p)k(p) + \frac{-k(p)}{h'(p) + \left(\tilde{N} - 1\right)f'(p)}[(\tilde{v} - p)h'(p) - h(p)]$$

where

$$h(p) + \left(\tilde{N} - 1\right)f(p) + \tilde{y} = 0. \tag{A6}$$

Equation (A6) defines a random variable $\tilde{p}$ and its joint distribution with the pair of random variables $(\tilde{y}, \tilde{N})$. In particular, the distribution of $\tilde{p}$ does not depend on the choice of the arbitrary function k. We can write

$$\begin{aligned} 0 &= E_i(\tilde{v} - \tilde{p})k(\tilde{p}) + \frac{-k(\tilde{p})}{h'(\tilde{p}) + \left(\tilde{N} - 1\right)f'(\tilde{p})}[(\tilde{v} - \tilde{p})h'(\tilde{p}) - h(\tilde{p})] \\ &= E\tilde{N}(\tilde{v} - p)k(p) + \tilde{N}\frac{-k(p)}{h'(p) + \left(\tilde{N} - 1\right)f'(p)}[(\tilde{v} - p)h'(p)h(p)] \end{aligned}$$

where for the last equality we use lemma 1.

In a symmetric equilibrium, we must have $h = f$. Hence, for an arbitrary function $k(p)$ we must have

$$0 = E\big(1 - \tilde{N}\big)(\tilde{v} - \tilde{p})k(\tilde{p}) + \frac{-k(\tilde{p})f(\tilde{p})}{f'(\tilde{p})}$$
$$= E\big(1 - \tilde{N}\big)[b(\tilde{y}) - \tilde{p}]k(\tilde{p}) + \frac{-k(\tilde{p})f(\tilde{p})}{f'(\tilde{p})}$$

where for the second equality, we use the law of iterated conditional expectation. Since the preceding equality should hold for every arbitrary function, $k(p)$, we must conclude that, in equilibrium,

$$0 = E\left\{\big(1 - \tilde{N}\big)[b(\tilde{y}) - \tilde{p}] - \frac{f(\tilde{p})}{f'(\tilde{p})}\,\middle|\, \tilde{p}\right\},$$

where the random variable $\tilde{p}$ is defined via $\tilde{N}f(p) + \tilde{y} = 0$.

LEMMA 3. For any function $h(N, q)$, such that $Eh(N, -y/N) < \infty$, we have

$$E[h(N, -y/N)\,|\,-y/N = q] = H(q)$$

where

$$H(q) = \frac{\sum_{n=1}^{K} h(n,\, q)ng(-nq)P_n}{\sum_{n=1}^{k} ng(-nq)P_n}, \tag{A7}$$

$g(y)$ is the density function of y, $P_n = \text{Prob}(N = n)$, and K is the upper bound on N.

Proof. The distribution of the pair $(\tilde{N}, -\tilde{y}/\tilde{N})$ is given by

$$\text{Prob}\left(\tilde{N} \le k,\, -\frac{\tilde{y}}{\tilde{N}} \le Q\right) = \sum_{n=1}^{k} P_n \int_{-\infty}^{Q} ng(-ng)dq$$

Indeed, for every integrable function $g(n,q)$, we have

$$Eh\left(\tilde{N}\frac{\tilde{y}}{\tilde{N}}\right) = \sum_n \int_{-\infty}^{\infty} g\left(n,\, -\frac{y}{n}\right)g(y)P_n dy = \sum_n \int_{-\infty}^{\infty} g(n,\, q)ng(-nq)P_n dq.$$

Now, take the function $g(n, q)$ to be the indicator function $I_{\{n \le k,\, q \le Q\}}$.

To verify the formula for conditional expectation, first note that, for functions that are independent of q, (A7) holds (see Durret 1966, p. 223). For general functions $h(n, q)$ that also depend on q, (A7) follows from the substitution rule: for a given q, $E[h(\tilde{N},\, -\tilde{y}/\tilde{N})\,|\,-\tilde{y}/\tilde{N} = q] = E[g_q(\tilde{N})\,|\,-\tilde{y}/\tilde{N} = q]$, where $g_q(\tilde{N}) = h(\tilde{N}, q)$.

Proof of Theorem 9

We provide here the missing parts of the proof. The l′Hospital rule implies that, in an open-book environment,

$$p_o'(0) = \lim_{q0} \frac{(N-1)[p_o(q) - b(-Nq)]}{q} = (N-1)[p'_o(0) + Nb'(0)]$$

Thus,

$$p'_o(0) = -\frac{(N-1)N}{N-2} b'(0).$$

In a closed-book environment,

$$p'_c(0) = \lim_{q0} \frac{h_1(q)p_c(q) - h_0(q)}{q} = p'_c(0)h_1(0) - h'_0(0).$$

Therefore,

$$p'_c(0) = -h'_0(q)/[1 + h_1(0)].$$

Now, $h'_0(q) = -b'(0)EN^2(N-1)/EN$ and $h_1(0) = EN(N-1)/EN$. We conclude

$$p'_c(0) = -\frac{EN^2(N-1)}{EN(N-2)} b'(0).$$

Thus,

$$\varphi_0(0) = E\frac{1}{N}\frac{(N-1)}{N-2} b'(0)$$
$$\varphi_c(0) = E\frac{1}{N}\frac{EN^2(N-1)}{EN(N-2)} b'(0)$$

Next, we need to show

$$E\frac{(N-1)}{N-2} < \frac{EN^2(N-1)}{EN(N-2)} E\frac{1}{N}.$$

Both terms $(N-2)/[N(N-1)]$ and $(N-1)/(N-2)$ are decreasing in N and hence have positive covariance.[23]

Hence,

$$E\frac{1}{N} = E\frac{1}{N}\frac{N-1}{N-1}\frac{N-1}{N-2} \geq E\frac{N-2}{N(N-1)} E\frac{N-1}{N-2}.$$

Also, the term $N^2(N-1)$ is increasing with N while $(N-2)/[N(N-1)]$ is decreasing. Consequently, the two terms have negative correlation and we get

$$EN(N-2) = EN^2(N-1)\frac{(N-2)}{N(N-1)} \leq EN^2(N-1)E\frac{(N-2)}{N(N-1)}.$$

23. Given a random variable N and two decreasing functions $f(N)$ and $g(N)$, we have $\mathrm{cov}(f, g) = \mathrm{cov}(f, g + \bar{f} - \bar{g})$, where $\bar{f} = Ef(N)$ and $\bar{g} = Eg(N)$. From the definition of covariance, it follows that the covariance is positive.

We conclude

$$EN^2(N-1)E\frac{1}{N} \geq EN^2(N-1)E\frac{N-2}{N(N-1)}E\frac{N-1}{N-2}EN \geq (N-2)E\frac{N-1}{N-2}.$$

To end the proof, we divide each side by the term $EN(N-2)$.

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